The Term Structure of Equity and Variance Risk Premia*

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Abstract

We study the term structure of variance swaps, equity and variance risk premia. A model-free analysis reveals a significant price jump component in variance swaps. A model-based analysis shows that investors' willingness to ensure against volatility risk increases after a market drop. This effect is stronger for short horizons, but more persistent for long horizons. During the financial crisis investors demanded large risk premia to hold equities, but the risk premia largely depended and strongly decreased with the holding horizon. The term structure of equity and variance risk premia responds differently to various economic factors.

Keywords: Variance Swap, Stochastic Volatility, Likelihood Approximation, Term Structure, Equity Risk Premium, Variance Risk Premium.

JEL Codes: C51, G12, G13.

1. Introduction

Over the last decade, the demand for volatility derivative products has grown exponentially, driven in part by the need to hedge volatility risk in portfolio management and derivative pricing. In 1993, the Chicago Board Options Exchange (CBOE) introduced the VIX as a volatility index computed as an average of the implied volatilities of short term, near the money, S&P100 options. Ten years later, the definition of the VIX was amended to become based on the more popular S&P500, itself the underlying of the most liquid index options (SPX), and to be computed in a largely model-free manner as a weighted average of option prices across all strikes at two nearby maturities, instead of relying on the Black–Scholes implied volatilities (e.g., Carr and Wu (2006).) Shortly thereafter, VIX futures and options on VIX were introduced at the CBOE Futures Exchange (CFE). Carr and Lee (2009) provide an excellent history of the market for volatility derivatives and a survey of the relevant methodologies for pricing and hedging volatility derivatives products.

Among volatility derivatives, variance swap (VS) contracts can be thought of as the basic building block. These are in principle simple contracts: the fixed leg agrees at inception that it will pay a fixed amount at maturity, the VS rate, in exchange to receiving a floating amount based on the realized variance of the underlying asset, usually measured as the sum of the squared daily log-returns, over the life of the swap. One potential difficulty lies in the path-dependency introduced by the realized variance.

The payoff of a VS can be replicated, under certain conditions, by dynamic trading in the underlying asset and a static position in vanilla options on that same underlying and maturity date. This insight, originally due to Neuberger (1994) and Dupire (1993), meant that the path-dependency implicit in VS could be circumvented; it also made possible an important literature devoted to analyzing and exploiting the various hedging errors when attempting to replicate a given VS (e.g., Carr and Madan (1998), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Jiang and Oomen (2008), Carr and Wu (2009), Carr and Lee (2010).) Because of the interest in replicating a given contract, VS rates have generally been studied at a single maturity.

But VS rates give rise naturally to a *term structure*, by varying the maturity at which the exchange of cash flows take place. The goal of this paper is to study the term structure of VS, equity and variance risk premia, to understand the implications for investors' perception of risk, and to determine which economic variables may drive these risk premia.

Studying the term structure of VS and risk premia is interesting for a number of reasons. First, the VS market is relatively unexplored and important event compared to option markets; indeed, the CBOE has listed new VS contracts since 2012.[1] Second, the term structure of VS provides *directly* market expectations about future volatility. This is in contrast for example to the option price surface that is affected by many factors. Thus, studying the term structure of VS should allow us to accurately estimate the term structure of variance risk premia. Third, equity and variance risk premia over different time horizons may respond to different economic indicators, uncovering the term structure of investors' perception of those risks. To investigate these aspects, we use actual, rather than synthetic, daily VS rates on the S&P500 index with fixed time to maturity of 2-, 3-, 6-, 12- and 24-month from January 4, 1996 to September 2, 2010.

We use a model-free method to assess the price jump component embedded in VS rates. Specifically, we compare VS rates and VIX-type indices extracted from options on the S&P500 index (SPX) for various maturities, using the CBOE and Carr and Wu (2009) methodologies. We find that a large and time-varying price jump component is embedded in VS rates, which becomes even more pronounced in the latter part of the sample. This indicates that either the price jump risk is heavily priced by VS traders or some segmentation between the VS and

[1] Since December 2012 the CBOE has listed new contracts called "S&P 500 Variance Futures." These are exchange-traded, marked-to-market variance swaps on the S&P500 with maturities ranging up to two years. See http://www.cfe.cboe.com/Products/Spec_VA.aspx.

option markets exits or both.

Various aspects of the VS term structure cannot be studied in a model-free manner, because the necessary data are either insufficient in quantity or simply unavailable.[2] To further the analysis of the VS term structure, we therefore rely on a parametric stochastic volatility model, namely a two-factor stochastic volatility model with price jumps and variance jumps, which is consistent with the salient empirical features of VS rates documented in the model-free analysis. The model is estimated using maximum-likelihood, combining time series information on stock returns with cross sectional information on the term structure of VS rates, thereby making inference in particular about risk premia theoretically sound.

Our model-based analysis shows that the integrated variance risk premium (IVRP), i.e., the ex-ante expected difference between objective and risk neutral integrated variance, is negative and usually exhibits a downward-sloping term structure. A negative risk premium implies that the VS holder is willing to pay a "large" premium, the VS rate, to get protection against volatility risk, which in turn induces a negative VS payoff on average at maturity. The downward sloping term structure means that the longer the maturity, the more negative the expected VS payoff. Moreover, after a volatility spike, investors' willingness to ensure against future volatility risk increases with the time horizon. This effect is stronger over short horizons (e.g., two months) but more persistent over long horizons (e.g., two years).

We also find that the term structure of the IVRP due to negative price jumps is negative, generally downward sloping in quiet times but upward sloping in turbulent times. Thus, the contribution of price jumps is modest in quiet times, but important during market drops, and mostly impacts the short-end of the IVRP term structure. This indicates that short-term variance risk premia mainly reflect investors' fear of a market drop, rather than the impact of

[2]For example, a model-free analysis of the term structure of jump risk in VS would require observations on long lived, out-of-the-money, SPX options with a fixed time to maturity. These options are, unfortunately, unavailable or at least not sufficiently liquid. Available options have discrete strike prices and fixed maturities, rather than fixed time to maturities. To carry out such a model-free analysis, interpolation and extrapolation schemes across strike prices and time to maturities are necessary with the potential to introduce significant approximation errors.

stochastic volatility on the investment set. This finding carries clear asset pricing implications. Equilibrium models seeking to explain asset returns and their volatility in the short run should feature price jumps, investors' aversion to jump risk, and intertemporal utility structures, in addition to stochastic volatility.

Next, we link the term structure of IVRP to economic indicators. Through regression analysis, we show that the term structure of IVRP responds to variables proxying for equity, option, corporate and Treasury bond market conditions. Not surprisingly, a drop in the S&P500 index induces a more negative IVRP, but this effect "quickly dies out" in the term structure of the IVRP, becoming statistically insignificant beyond a 6-month horizon. In other words, daily changes of the S&P500 index strongly impact investors' perception of volatility risk, but only over short horizons. Similarly, an increase of corporate credit riskiness increases the IVRP in absolute value but only over relatively short horizons (up to six months). This suggests that VS market participants view this phenomenon as being transient in terms of its impact on volatility risk. The VIX index, despite being a 30-day volatility index, has a fairly uniform and strong impact throughout the term structure of the IVRP, acting more like a "level factor," rather than a short-term factor, for variance risk premia.

In analogy to the term structure of IVRP, we also study the term structure of the equity risk premium. We define the integrated equity risk premium, IERP, as the ex-ante expected excess return from buying and holding the S&P500 index over a fixed time horizon, such as 2-month or 1-year. We find that equity risk premia are strongly countercyclical, and become large and positive during crisis times. The term structure of IERP is slightly upward sloping in quiet times but steeply downward sloping during market crashes. This indicates that during a financial crisis investors demand large risk premia to hold risky stocks, but the risk premia largely depend and strongly decrease with the holding horizon. For example, in Fall 2008, after Lehman Brothers' bankruptcy, our estimates of 2-month equity risk premia reached historically high values, around 50%. During average volatility periods, equity risk premia are approximately 6.5%, in line with historical estimates.

Finally, as for the IVRP, we conduct regression analysis to understand which economic variables may drive the term structure of the IERP. We find that an increase in the VIX index increases the IERP, but the longer the time horizon the smaller the effect. Hence, in contrast to the IVRP, the VIX index does not behave like a level factor for the IERP. Indicators of corporate credit riskiness have a positive and decreasing impact on the term structure of the IERP. This suggests that distress conditions of the corporate sector exacerbate the countercyclical variation of the IERP, but only for the IERP over short horizons (up to six months). Other variables impact the slope of the IERP term structure. For example, the slope of the yield curve, which increased significantly in Fall 2008, has a positive impact on the short-end and a negative impact on the long-end of the IERP term structure. To the extent that the slope of the yield curve reflects "flight-to-liquidity", investors' selling pressure of equities (to increase their allocations to treasuries) appears to increase the IERP over short horizons. Investors also seem to anticipate that they will rebalance their portfolios from treasuries to equities when the crisis will be over, and thus the negative impact of the slope of the yield curve on IERP over long horizons (e.g., one year). All in all, our empirical findings point to a rich impact of economic indicators throughout the term structure of equity and variance risk premia.

This paper is related to various strands of the literature. A number of studies have estimated stochastic volatility models to recover risk premia; see, e.g., Bakshi et al. (1997), Pan (2002), Broadie et al. (2007) and references therein. However, most of these studies fit stochastic volatility models to option prices and analyze instantaneous risk premia. We consider a different important market and analyze risk premia over fixed time horizons, namely the term structure of risk premia. While any stochastic volatility model has implications for the term structure of risk premia, models fitted directly to the term structure of VS should benefit from the VS rates being the term structure of variance risk.

A fast growing literature has been focusing on the variance risk premium, albeit almost exclusively on a single maturity. Bollerslev et al. (2009) linked the one-month variance risk premium to time-varying economic uncertainty and show empirically that this premium predicts aggregate market returns. Bekaert and Hoerova (2014) expand the evidence on the predictive power of one-month variance risk premium for stock returns. Carr and Wu (2009), Bollerslev and Todorov (2011) and others provide model-free analysis of a single maturity variance risk premium. Mueller et al. (2013) study the term structure of Treasury bond variance risk premia and document a significant negative risk premium, albeit approaching zero when the time horizon increases. Recent studies investigate VS contracts. For example Amengual (2008) studies the term structure of S&P500 variance risk premia, under the assumption that the jump risk premium is zero. Dew-Becker et al. (2014) investigate the term structure of zero-coupon VS claims. Egloff et al. (2010) and Filipović et al. (2015) study optimal investment in VS contracts. We complement these studies by analyzing the term structure of variance risk premia, and linking these risk premia to economic indicators.

Several studies have analyzed the equity risk premium and the associated "puzzle", but mainly focusing on a *single* horizon (e.g., one year) and relying on *ex-post* market returns; see, e.g., Mehra (2006) for a review. We study the term structure of the ex-ante equity risk premia. Recently, van Binsbergen et al. (2013) and Martin (2013) provide related studies on equity risk premia, using different datasets and methods, and they also document large swings in equity risk premia, comparable to those we document here. We complement these studies by analyzing the term structure of equity risk premia and their economic drivers.

The structure of the paper is as follows. Section 2 briefly describes variance swaps and their properties. Section 3 introduces the model and estimation methodology. Section 4 presents the actual estimates. Section 5 reports risk premium estimates. Section 6 concludes. The Appendix contains technical derivations.

2. Variance Swaps

We introduce the general setup we will work with in order to analyze the term structure of VS contracts. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying usual conditions (e.g., Protter (2004)), with P denoting the objective or historical probability measure. Let S be a semimartingale modeling the stock (or index) price process with dynamics

$$dS_t/S_{t\_} = \mu_t\, dt + \sqrt{v_t}\, d\tilde{W}_t^P + (\exp(J_t^{s,P}) - 1)\, dN_t^P - \nu_t^P\, dt \tag{1}$$

where μ_t is the drift, v_t the spot variance, $\tilde{W}_t^P$ a Brownian motion, N_t^P a counting jump process with stochastic intensity λ_t^P, $J_t^{s,P}$ the random price jump size, and $\nu_t^P = g_t^P \lambda_t^P$ the compensator with $g_t^P = E_t^P[\exp(J^s) - 1]$ and E_t^P the time-t conditional expectation under P. When a jump occurs, the induced price change is $(S_t - S_{t\_})/S_{t\_} = \exp(J_t^{s,P}) - 1$, which implies that $\log(S_t/S_{t\_}) = J_t^{s,P}$. Thus, $J_t^{s,P}$ is the random jump size of the log-price under P. When no confusion arises superscripts and subscripts are omitted. The dynamics of the drift, variance, and jump component are left unspecified and in this sense the first part of the analysis of VS contracts will be *model-free*. Indeed, the Model (1) subsumes virtually all models used in finance with finite jump activity.

Let $t = t_0 < t_1 < \cdots < t_n = t + \tau$ denote the trading days over a given time period $[t, t+\tau]$, for e.g., six months. The typical convention employed in the market is for the floating leg of the swap to pay at $t + \tau$ the annualized realized variance defined as the annualized sum of daily squared log-returns (typically closing prices) over the time horizon $[t, t+\tau]$:

$$\mathrm{RV}_{t,t+\tau} = \frac{252}{n} \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 . \tag{2}$$

Like any swap, no cash flow changes hands at inception of the contract at time t; the fixed leg of the VS agrees to pay an amount fixed at time t, defined as the VS rate, $\mathrm{VS}_{t,t+\tau}$. Any payment takes place in arrears. Unlike many other swaps, such as interest rates or currency swaps, a VS does not lead to a repeated exchange of cash flows, but rather to a single one at expiration, at time $t+\tau$. Therefore, at maturity, $t+\tau$, the long position in a VS contract receives the difference between the realized variance between times t and $t+\tau$, $\mathrm{RV}_{t,t+\tau}$, and the VS rate, $\mathrm{VS}_{t,t+\tau}$, which was fixed at time t. The difference is multiplied by a fixed notional amount to convert the payoff to dollar terms:

$$(\mathrm{RV}_{t,t+\tau} - \mathrm{VS}_{t,t+\tau}) \times (\text{notional amount}).$$

If the time period $[t, t+\tau]$ will be an unexpected high volatility period, then the realized variance $\mathrm{RV}_{t,t+\tau}$ will be higher than the VS rate $\mathrm{VS}_{t,t+\tau}$ set at time t, which in turn will trigger a positive payoff to the long side of the contract. Thus, variance swaps are effectively insurance contracts against high volatility.

The analysis of VS contracts is simplified when the realized variance is replaced by the quadratic variation of the log-price process. It is well-known that when $\sup_{i=1,\ldots,n}(t_i - t_{i-1}) \to 0$ the realized variance in Equation (2) converges in probability to the annualized quadratic variation of the log-price, $\mathrm{QV}_{t,t+\tau}$, (e.g., Jacod and Protter (1998)):

$$\frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 \longrightarrow \frac{1}{\tau}\int_t^{t+\tau} v_u\,du + \frac{1}{\tau}\sum_{u=N_t}^{N_{t+\tau}} (J_u^s)^2 = \mathrm{QV}^c_{t,t+\tau} + \mathrm{QV}^j_{t,t+\tau} = \mathrm{QV}_{t,t+\tau} \qquad (3)$$

which is itself the sum of two terms, one due to the continuous part of the Model (1), $\mathrm{QV}^c_{t,t+\tau}$, and one to its discontinuous or jump part, $\mathrm{QV}^j_{t,t+\tau}$. This approximation is commonly adopted in practice and is quite accurate at the daily sampling frequency (e.g., Broadie and Jain (2008) and Jarrow et al. (2013)), as is the case in our dataset. Market microstructure noise, while generally an important concern in high frequency inference, is largely a non-issue at the level of daily returns.

As usual, we assume absence of arbitrage, which implies the existence of an equivalent risk neutral measure Q. By convention, the VS contract has zero value at inception. Assuming that the interest rate does not depend on the quadratic variation, which is certainly a tenuous assumption and one commonly made when valuing these contracts, no arbitrage implies that the VS rate is

$$\mathrm{VS}_{t,t+\tau} = E_t^Q[\mathrm{QV}_{t,t+\tau}] = \overline{v}_{t,t+\tau}^Q + E_t^Q[(J^s)^2]\,\overline{\lambda}_{t,t+\tau}^Q \tag{4}$$

where E_t^Q denotes the time-t conditional expectation under Q, $\overline{v}_{t,t+\tau}^Q = E_t^Q[\mathrm{QV}_{t,t+\tau}^c]$, and $\overline{\lambda}_{t,t+\tau}^Q = E_t^Q \int_t^{t+\tau} \lambda_u^Q \, du/\tau$, i.e., the average risk neutral jump intensity.

The VS rate depends, of course, on the information available at time t. It also depends on the time to maturity, τ. The latter dependence produces the term structure we are interested in.

2.1. *Preliminary Data Analysis*

Our dataset consists of over the counter quotes on VS rates on the S&P500 index provided by a major broker-dealer in New York City. The data are daily closing quotes on VS rates with fixed time to maturities of 2, 3, 6, 12, and 24 months from January 4, 1996 to September 2, 2010, resulting in 3,624 observations for each maturity. Standard statistical tests do not detect any day-of-the-week effect, so we use all available daily data.

We start by identifying some of the main features of the VS rates data. Figure 1 shows the term structure of VS rates over time. VS rates appear to be mean-reverting, volatile, with spikes and clustering during the major financial crises over the last 15 years, and historically high values during the acute phase of the recent financial crisis in Fall 2008. While most term structures are upward sloping (53% of our sample), they are often $\cup$-shape too (23% of our sample). The remaining term structures are roughly split in downward sloping and $\cap$-shape term structures.[3]

[3] On some occasions, the term structure is $\sim$-shape, but the differences between, for e.g., the 2 and 3 months VS rates are virtually zero and these term structures are nearly $\cup$-shape.

The bottom and peak of the ∪- and ∩-shape term structures, respectively, can be anywhere at 3 or 6 or 12 months to maturity VS rate. The slope of the term structure (measured as the difference between the 24 and 2 months VS rates) shows a strong negative association with the contemporaneous volatility level. Thus, in high volatility periods or turbulent times, the short-end of the term structure (VS rates with 2 or 3 months to maturity) rises more than the long-end, producing downward sloping term structures.

Tables 1 and 2 provide summary statistics of our data. For the sake of interpretability, we follow market practice and report VS rates in volatility percentage units, i.e., $\sqrt{\mathrm{VS}_{t,t+\tau}} \times 100$. Various patterns emerge from these statistics. The mean level and first order autocorrelation of swap rates are slightly but strictly increasing with time to maturity. The standard deviation, skewness and kurtosis of swap rates are strictly decreasing with time to maturity. Ljung–Box tests strongly reject the hypothesis of zero autocorrelations, while generally Dickey–Fuller tests do not detect unit roots,[4] except for longest maturities – it is well-known that the outcome of standard unit root tests should be carefully interpreted with slowly decaying memory processes; e.g., Schwert (1987). First order autocorrelations of swap rates range between 0.982 and 0.995, confirming mean reversion in these series. As these coefficients increase with time to maturity, the longer the maturity the higher the persistence of VS rates with mean half-life[5] of shocks between 38 and 138 days. Daily changes in VS rates are on average close to zero, non-normal, and exhibit far less persistence than VS rates in levels.

Principal Component Analysis (PCA) shows that the first principal component explains about 95.4% of the total variance of VS rates and can be interpreted as a level factor, while the second principal component explains an additional 4.4% and can be interpreted as a slope factor.[6] This finding is somehow expected because PCA of several other term structures, such as bond yields, produce qualitatively similar results. Less expected is that two factors explain

[4]Under the null hypothesis of unit root the Dickey–Fuller test statistic has zero expectation.
[5]The half-life H is defined as the time necessary to halve a unit shock and solves $\varrho^H = 0.5$, where ϱ is the first order autocorrelation coefficient.
[6]To save space, factor loadings are not reported, but are available from the authors upon request.

nearly all the variance of VS rates, i.e., 99.8%. Repeating the PCA for various subsamples produces little variation in the first two factors and explained total variance. Overall, PCA suggests that at most two factors are driving VS rates. When compared to typical term structures of bond yields, the one of VS rates appears to be simpler, as a third principal component capturing the curvature of the term structure is largely nonexistent here.

Table 1, Panel D, also shows summary statistics of ex-post realized variance of S&P500 index returns for various time to maturities. Realized variances are substantially lower on average than VS rates. Hence, shorting variance swaps is profitable on average. However, realized variances are also more volatile, positively skewed and leptokurtic than VS rates, which highlights the riskiness of shorting VS contracts. The large variability and in particular the positive skewness of ex-post realized variances can induce large losses to the short side of the contract. The ex-post variance risk premium, i.e., the difference between average realized variance and VS rate, is negative and increasing with time to maturities. Thus, shorting long-term variance swaps is on average more profitable than shorting short-term variance swaps.

2.2. *Model-free Jump Component in Variance Swap Rates*

We now provide a model-free assessment of the price jump component in VS rates by taking advantage of recent theoretical advances.[7] Under certain conditions, if the stock price process is continuous, the VS payoff can be replicated by dynamic trading in futures contracts (or in the underlying asset) and a static position in a continuum of European options with different strikes and same maturity. The replication is model-free in the sense that the stock price can follow the general Model (1), but with the restriction $\lambda_t^P = 0$ and/or $J_t^{s,P} = 0$.

If the stock price has a jump component, this replication no longer holds. This observation makes it possible to assess whether VS rates embed a priced jump component and to quantify

[7]See Neuberger (1994), Dupire (1993), Carr and Madan (1998), Demeterfi et al. (1999), Britten-Jones and Neuberger (2000), Jiang and Tian (2005), Jiang and Oomen (2008), Carr and Wu (2009), Carr and Lee (2010) and Fuertes and Papanicolaou (2012).

how large it is. In practice, of course, only a typically small number of options is available to construct the replicating portfolio for a given horizon τ. Moreover, options are available only for a few maturities that typically do not match the horizon τ. An interpolation across maturities is therefore necessary. Jiang and Tian (2005) provide a detailed discussion of the issues that introduce approximation errors. Our findings below should be interpreted keeping in mind these interpolation errors.

Our procedure to detect the price jump component in VS rates is as follows. Model (1) implies the following risk neutral dynamic for the futures price F_t

$$d\log F_t = -\frac{1}{2}v_t\,dt + \sqrt{v_t}dW_t^Q + J_t^{s,Q}\,dN_t^Q - E_t^Q[\exp(J^s)-1]\lambda_t^Q\,dt.$$

The (squared) VIX index is obtained from an options portfolio that replicates a log contract[8]

$$\text{VIX}_{t,t+\tau} = -\frac{2}{\tau}E_t^Q\left[\log\frac{F_{t+\tau}}{F_t}\right] = -\frac{2}{\tau}E_t^Q\int_t^{t+\tau} d\log F_u = \overline{v}_{t,t+\tau}^Q + 2E_t^Q[\exp(J^s)-1-J^s]\,\overline{\lambda}_{t,t+\tau}^Q.$$

The difference between the VS rate in (4) and $\text{VIX}_{t,t+\tau}$ is

$$\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau} = 2E_t^Q\left[\frac{(J^s)^2}{2} + J^s + 1 - \exp(J^s)\right]\overline{\lambda}_{t,t+\tau}^Q. \tag{5}$$

Up to a discretization error, $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is a model-free assessment of the price jump term in the right hand side. If the price jump is zero, i.e., $J^s = 0$ and/or the intensity $\overline{\lambda}_{t,t+\tau}^Q = 0$, then $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is zero as well, and the VIX index is indeed a VS rate. If the price jump is not zero, then $\text{VS}_{t,t+\tau} - \text{VIX}_{t,t+\tau}$ is expected to be positive. The reason is that the

[8]The identity

$$\frac{F_{t+\tau}}{F_t} - 1 - \log\frac{F_{t+\tau}}{F_t} = \int_0^{F_t}\frac{(K-F_{t+\tau})^+}{K^2}dK + \int_{F_t}^{\infty}\frac{(F_{t+\tau}-K)^+}{K^2}dK$$

leads to computing the VIX index using forward prices of the out-of-the-money put and call options on the S&P500 index with maturity $t+\tau$. The VIX index is based on a calendar day counting convention and linear interpolation of options whose maturities straddle 30 days (e.g., Carr and Wu (2006) provide a description of the VIX calculation.)

function in the square brackets in Equation (5) is downward sloping and passing through the origin. If the jump distribution under Q is mainly concentrated on negative values, suggesting that jump risk is priced, the expectation in Equation (5) tends to be positive.[9] The average risk neutral jump intensity $\overline{\lambda}^Q_{t,t+\tau}$ is always nonnegative. Note that even if the jump risk is not priced, i.e., the jump size distributions under P and Q are the same, $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ could still be nonzero.

Following the revised post-2003 VIX methodology, we calculate daily VIX-type indices, $\mathrm{VIX}_{t,t+\tau}$, for $\tau = 2$, 3, and 6 months to maturity from January 4, 1996 to September 2, 2010 and compute the difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. SPX option prices are obtained from Option-Metrics. Although it is straightforward to calculate VIX-type indices for longer maturities, the interpolation of existing maturities straddling 12 and 24 months is likely to introduce significant approximation errors.

Table 1, Panel B, shows summary statistics of calculated VIX-type indices. These indices have the same term structure features as VS rates, qualitatively. However, on average, VS rates are higher, more volatile, skewed, and leptokurtic than VIX-type indices for each maturity. Moreover, the difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ increases with time to maturity. Figure 2 shows time series plots of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ for the various times to maturity. Such differences are mostly positive, statistically significant, larger during market turmoils but sizeable also in quiet times. A positive difference is not a crisis-only phenomenon, when jumps in stock price are more likely to occur and investors may care more about jump risk. Despite the interpolation errors mentioned above, we conclude that these findings are consistent with the presence of a significant jump component embedded in VS rates.

A few reasons are conceivable for a non-zero difference of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. The first reason can be that, since European options on the S&P500 index (SPX) are more liquid than

[9] Up to a third order Taylor expansion, the expectation in Equation (5) is proportional to $-E^Q_t\left[(J^s)^3\right]$. If price jumps exhibit negative skewness under Q, then $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ is again expected to be positive.

VS contracts, a larger liquidity risk premium is embedded in VS rates than in SPX options. Everything else equal, the higher the illiquidity of VS the higher the return of a long position in VS should be, reflecting a liquidity risk premium. However, this would imply that the higher the liquidity risk premium, the lower the VS rate. Thus, if anything, liquidity issues should bias downward, an otherwise larger and positive difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$.

A second reason for the non-zero difference in (5) can be that the SPX and VS are segmented or disconnected markets. In that case, comparing asset prices from the two markets can easily generate large gaps between $\mathrm{VS}_{t,t+\tau}$ and $\mathrm{VIX}_{t,t+\tau}$. On one hand, there is anecdotal evidence that VS contracts are typically hedged with SPX options and vice versa.[10] Thus, it is unlikely that the two markets are completely segmented. On the other hand, Bardgett et al. (2014) provide evidence that VIX derivatives and SPX options carry conflicting information about volatility dynamics, which can be interpreted as a form of segmentation between volatility and option markets. A temporary disconnection between the two markets could explain the negative difference $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$ observed on a few occasions in Fall 2008. For example if the SPX market reacts more quickly than the VS market to negative news, option prices increase faster than VS rates, inducing a negative difference.

While a positive difference in (5) is economically sensible, the remaining question is whether quantitatively the difference documented in Table 1 is economically "fair." To tackle this issue, we computed the difference in (5) using the stochastic volatility Model (6)–(7), as well as other models estimated in the literature. Although these models can produce a positive and time-varying difference, they cannot match the observed large time-variation of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$. Therefore, based on this metric, the positive difference appears to be excessively high, hinting to some segmentation between the VS and SPX markets.

The CBOE methodology to select options for the VIX calculation is to include all out-of-the-

[10]The difficulties involved in carrying out such hedging strategies became prominent in October 2008 when volatility reached historically high values (see Schultes (2008).)

money options, far in the moneyness range, until two consecutive zero bid prices are found. The rationale is to exclude illiquid options from the VIX calculation. Unfortunately, this procedure implies that the actual number of options used in the VIX calculation can change substantially from one day to the next, for example if options with zero bid price are suddenly traded and deeper out-of-the-money options had non-zero bid prices. This may produce some instabilities in the calculated VIX-type indices.[11]

To address this issue, we also calculated the VIX-type indices using the Carr and Wu (2009) methodology.[12] Table 1, Panel C, shows that the corresponding VIX-type indices are on average rather constant across maturities and closer to the VS rates than VIX-type indices based on the CBOE methodology. VIX-type indices based on the Carr–Wu methodology are still less volatile and somewhat smaller than VS rates for the 6-month time to maturity (and even more so for the unreported 12-month time to maturity). The corresponding time series of $\mathrm{VS}_{t,t+\tau} - \mathrm{VIX}_{t,t+\tau}$, for $\tau = 2$, 3, 6 months, are similar to the trajectories shown in Figure 2 and exhibit a significant time variation. All in all, based on the Carr–Wu methodology, the VS market appears to generate VS rates which are roughly in line with option market's expectations of future quadratic variations, at least over short time horizons. There is however an important difference between the CBOE and Carr–Wu methodologies, namely that only the former can be associated to an actual trading strategy, as it only involves traded options. Therefore, considering only tradable assets, the difference between VS and VIX-type indices appears to be

[11] Andersen et al. (2015) argue that the CBOE rule for selecting liquid options induces large instabilities in the intraday calculation of the VIX index, especially during periods of market turmoil, when an accurate assessment of volatility risk is most needed. We use the CBOE methodology to compute VIX-type indices on a daily basis. These instabilities should be less severe than on an intraday basis.

[12] The Carr–Wu methodology is as follows. For a given day t and time to maturity τ, implied volatilities at different moneyness levels are linearly interpolated to obtain 2,000 implied volatility points. The strike range is ± 8 standard deviations from the current stock price. The standard deviation is approximated by the average implied volatility. For moneyness below (above) the lowest (highest) available moneyness level in the market, the implied volatility at the lowest (highest) strike price is used. Given the interpolated implied volatilities, the forward price at day t of out-of-the-money options with different strikes K and time to maturity τ, $O_t(K,\tau)$, are computed using the Black–Scholes formula. The VIX-type index is then given by a discretization of $2/\tau \int_0^\infty O_t(K,\tau)/K^2\, dK$. This procedure is repeated for each day t in our sample and for the two time to maturities available in the market, say $\underline{\tau}$ and $\overline{\tau}$, straddling the time to maturity τ (which may not be available in the market), i.e., $\underline{\tau} \leq \tau \leq \overline{\tau}$, where $\tau = 2$, 3, 6 months. Finally, the linear interpolation across time to maturities of $2/\underline{\tau} \int_0^\infty O_t(K,\underline{\tau})/K^2\, dK$ and $2/\overline{\tau} \int_0^\infty O_t(K,\overline{\tau})/K^2\, dK$ gives the (squared) VIX-type index for the time to maturity τ.

substantial.

2.3. A Parametric Stochastic Volatility Model

The limitations of the data available make it necessary to adopt a parametric structure, with a specification informed by the model-free analysis above, in order to go further. So we now parameterize the Model (1). Given the data analysis above, as well as the evidence in Gatheral (2008) and Egloff et al. (2010) that two factors are both necessary and sufficient to accurately capture the dynamics of the VS rates, we adopt under the objective probability measure P, the following model for the ex-dividend stock price and its variance:

$$\begin{aligned} dS_t/S_{t_-} &= \mu_t\, dt + \sqrt{(1-\rho^2)v_t}\, dW_{1t}^P + \rho\sqrt{v_t}\, dW_{2t}^P + (\exp(J_t^{s,P}) - 1)\, dN_t - \nu_t^P\, dt \\ dv_t &= k_v^P(m_t\, k_v^Q/k_v^P - v_t)\, dt + \sigma_v\sqrt{v_t}\, dW_{2t}^P + J_t^{v,P}\, dN_t \\ dm_t &= k_m^P(\theta_m^P - m_t)\, dt + \sigma_m\sqrt{m_t}\, dW_{3t}^P \end{aligned} \tag{6}$$

where $\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + (g^P - g^Q)\lambda_t$, r is the risk free rate and δ the dividend yield, both taken to be constant for simplicity only. The instantaneous correlation between stock returns and spot variance changes, ρ, captures the so-called leverage effect. The base Brownian increments, $dW_{it}^P, i = 1, 2, 3$, are uncorrelated.[13]

The random price jump size, $J_t^{s,P}$, is independent of the filtration generated by the Brownian motions and jump process, and normally distributed with mean μ_j^P and variance σ_j^2. Hence, $g^P = \exp(\mu_j^P + \sigma_j^2/2) - 1$ is the Laplace transform of the random jump size. Similarly, $g^Q = \exp(\mu_j^Q + \sigma_j^2/2) - 1$. The counting process N_t has the same jump intensity under the P and Q measures, and it is given by $\lambda_t = \lambda_0 + \lambda_1 v_t$, where λ_0 and λ_1 are positive constants. This specification allows for more jumps to occur during more volatile periods, with the intensity bounded away from 0 by λ_0. Bates (2006) provides time series evidence that the jump intensity

[13]Under this model specification, $d\tilde{W}_t^P$ in Model (1) becomes $\sqrt{(1-\rho^2)}\, dW_{1t}^P + \rho\, dW_{2t}^P$ in Model (6).

is stochastic. Besides the empirical evidence on jumps in stock returns, the main motivation for introducing such a jump component in stock returns is to account for the jump component in VS rates, as suggested by our model-free analysis in Section 2.2.

The spot variance, v_t, follows a two-factor model where m_t controls its stochastic long-run mean or central tendency. The speed of mean reversion is k_v^P under P, k_v^Q under Q and $k_v^P = k_v^Q - \gamma_2\sigma_v$, where γ_2 is the risk premium for W_{2t}^P; Section 2.4 discusses the last equality. The process m_t controlling the stochastic long run mean follows its own stochastic mean reverting process and mean reverts to a positive constant θ_m^P, when the speed of mean reversion k_m^P is positive. Typically, v_t is fast mean reverting and volatile to capture sudden movements in volatility, while m_t is more persistent and less volatile to capture long term movements in volatility. Several studies provide evidence that two factors are necessary to describe variance dynamics.[14] The square-root specification of the diffusion components, $\sigma_v\sqrt{v_t}$ and $\sigma_m\sqrt{m_t}$, is adopted to keep Model (6) close to commonly used models, e.g., Chernov and Ghysels (2000), Pan (2002), Broadie et al. (2007, 2009), Egloff et al. (2010), and Todorov (2010).

The random jump size of the spot variance, $J_t^{v,P}$, is independent of W_t^P and $J_t^{s,P}$, and exponentially distributed with parameter μ_v^P, i.e., $E^P[J_t^v] = \mu_v^P$, ensuring that v_t stays positive. Thus, the variance jump $J_t^{v,P}$ captures quick upward movements of v_t. The Model (6) features contemporaneous jumps both in returns and variance, that is the double-jump model introduced by Duffie et al. (2000). Eraker et al. (2003) fit models with contemporaneous and independent jumps in returns and variance to S&P500 data. They find that the two models perform similarly, but the model with contemporaneous jumps is estimated more precisely. Eraker (2004), Broadie et al. (2007), Chernov et al. (2003), and Todorov (2010) provide further evidence for contemporaneous jumps in returns and variance.

[14]These studies include Andersen et al. (2002), Alizadeh et al. (2002), Adrian and Rosenberg (2008), Engle and Rangel (2008), Christoffersen et al. (2009) and Corradi et al. (2013).

Model (6) covers existing stochastic volatility models along most dimensions. For example, none of the studies cited above allow at the same time for stochastic long run mean, stochastic jump intensity and jumps in returns and variance. Bakshi et al. (1997), Bates (2000, 2006), Pan (2002), Eraker et al. (2003), Eraker (2004), Broadie et al. (2007, 2009) set m_t to a constant positive value. Almost all studies assume either constant jump intensities (e.g., Eraker et al. (2003) and Broadie et al. (2007)) or jumps in returns but not in variance (e.g., Pan (2002) and Broadie et al. (2009)).

Under Q, the ex-dividend price process evolves as

$$\begin{aligned} dS_t/S_{t_-} &= (r-\delta)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^Q + \rho\sqrt{v_t}\,dW_{2t}^Q + (\exp(J_t^{s,Q})-1)\,dN_t - \nu_t^Q\,dt \\ dv_t &= k_v^Q(m_t - v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_{2t}^Q + J_t^{v,Q}\,dN_t \\ dm_t &= k_m^Q(\theta_m^Q - m_t)\,dt + \sigma_m\sqrt{m_t}\,dW_{3t}^Q \end{aligned} \tag{7}$$

where the Brownian motions $W_i^Q, i = 1,2,3$, price jump size $J^{s,Q}$, counting jump process N, its compensator ν^Q, and variance jump size $J^{v,Q}$ are governed by the measure Q.

Given the stochastic volatility model above, the VS rate is available in closed form. We first calculate $\overline{v}_{t,t+\tau}^Q$ in Equation (4). Interchanging expectation and integration (justified by Tonelli's theorem)

$$\overline{v}_{t,t+\tau}^Q = \frac{1}{\tau}\int_t^{t+\tau} E_t^Q[v_u]\,du = (1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q + \phi_v^Q(\tau)v_t + \phi_m^Q(\tau)\tilde{m}_t \tag{8}$$

where $\tilde{m}_t = (k_v^Q m_t + \mu_v^Q\lambda_0)/\tilde{k}_v^Q$, $\tilde{k}_v^Q = k_v^Q - \mu_v^Q\lambda_1$, and

$$\begin{aligned} \phi_v^Q(\tau) &= \left(1-\exp(-\tilde{k}_v^Q\tau)\right)/(\tilde{k}_v^Q\tau) \\ \phi_m^Q(\tau) &= \left(1+\exp(-\tilde{k}_v^Q\tau)k_m^Q/(\tilde{k}_v^Q-k_m^Q) - \exp(-k_m^Q\tau)\tilde{k}_v^Q/(\tilde{k}_v^Q-k_m^Q)\right)/(k_m^Q\tau). \end{aligned}$$

Equation (8) is obtained using the risk neutral jump-compensated dynamic of v_t.[15] Finally, using independence among $J^{s,Q}$, $J^{v,Q}$ and N

$$\text{VS}_{t,t+\tau} = \overline{v}^Q_{t,t+\tau} + E^Q_t[(J^s)^2]\,\overline{\lambda}^Q_{t,t+\tau} \tag{9}$$

where $E^Q_t[(J^s)^2] = E^Q[(J^s)^2] = (\mu^Q_j)^2 + \sigma^2_j$, as the return jump size is time-homogeneous, and $\overline{\lambda}^Q_{t,t+\tau} = \lambda_0 + \lambda_1 \overline{v}^Q_{t,t+\tau}$. Note that if the variance jump component was absent, i.e., $J^{v,Q}_t = 0$, then $\mu^Q_v = 0$ and $\overline{v}^Q_{t,t+\tau}$ had the same analytical expression as in (8) with $\tilde{m}_t = m_t$ and $\tilde{k}^Q_v = k^Q_v$.

Given the linearity of the VS payoff in the spot variance, only the drift of v_t enters the VS rate. The martingale part of v_t (diffusion and jump compensated parts) affects only the dynamic of $\text{VS}_{t,t+\tau}$. The Q-expectation of the stochastic jump intensity provides a time-varying contribution to $\text{VS}_{t,t+\tau}$, given by $\overline{\lambda}^Q_{t,t+\tau}$, which depends on the time to maturity of the contract.

2.4. *Market Prices of Risk*

As in Pan (2002), Aït-Sahalia and Kimmel (2010), and others, we specify the market price of risks for the Brownian motions as

$$\Lambda'_t = [\gamma_1 \sqrt{(1-\rho^2)v_t}, \quad \gamma_2 \sqrt{v_t}, \quad \gamma_3 \sqrt{m_t}] \tag{10}$$

where $'$ denotes transposition. Thus, P and Q parameters controlling v_t and m_t are related as follows

$$k^P_v = k^Q_v - \gamma_2 \sigma_v, \quad k^P_m = k^Q_m - \gamma_3 \sigma_m, \quad \theta^P_m = \theta^Q_m\, k^Q_m / k^P_m.$$

[15]The risk neutral jump-compensated dynamic is $dv_t = k^Q_v(m_t - v_t)\,dt + \mu^Q_v(\lambda_0 + \lambda_1 v_t)dt + dM^Q_t$, where the Q-martingale increment $dM^Q_t = \sigma_v \sqrt{v_t}\,dW^Q_{2t} + J^{v,Q}_t\,dN_t - \mu^Q_v(\lambda_0 + \lambda_1 v_t)dt$. Rewriting the dynamic as $dv_t = \tilde{k}^Q_v(\tilde{m}_t - v_t)\,dt + dM^Q_t$ gives the expressions for $\tilde{k}^Q_v$ and $\tilde{m}_t$. Applying Itô's Lemma to $e^{\tilde{k}^Q_v t} v_t$, integrating between time t and s, and rearranging terms, as usual, give

$$v_s = v_t e^{-\tilde{k}^Q_v (s-t)} + \int_t^s e^{-\tilde{k}^Q_v (s-u)} \tilde{k}^Q_v \tilde{m}_u\,du + \int_t^s e^{-\tilde{k}^Q_v (s-u)} dM^Q_u.$$

Taking E^Q_t, the last term above vanishes. The expectation $E^Q_t[\tilde{m}_u]$ can be computed following similar steps. Calculating all integrals gives Equation (8).

More flexible specifications of the market price of risks for the Brownian motions have been suggested (e.g., Cheridito et al. (2007).) In the present application, there does not appear to be a strong need for an extension of (10), given the tradeoffs between the benefits of a more richly parameterized model and the costs involved in its estimation and out-of-sample performance.

The price jump size risk premium is $(g^P - g^Q) = \exp(\mu_j^P + \sigma_j^2/2) - \exp(\mu_j^Q + \sigma_j^2/2)$. The variance of the price jump size is the same under P and Q, implying that the jump distribution has the same shape but potentially different location under P and Q. As, e.g., in Pan (2002), Eraker (2004), and Broadie et al. (2007), we assume that the jump intensity is the same under both measures. The main motivation for this assumption is the well-known limited ability to estimate jump components in stock returns and the corresponding risk premium using daily data. Thus, all price jump risk premium is absorbed by the price jump size risk premium, $(g^P - g^Q)$. The total price jump risk premium is time-varying and given by $(g^P - g^Q)(\lambda_0 + \lambda_1 v_t)$. Similarly, the variance jump premium is $(\mu_v^P - \mu_v^Q)(\lambda_0 + \lambda_1 v_t)$.

The jump component makes the market incomplete with respect to the risk free bank account, the stock and any finite number of derivatives. Hence, the state price density is not unique. The specification we adopt is

$$\begin{aligned}\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} &= \exp\left(-\int_0^t \Lambda_s' \, dW_s^P - \frac{1}{2}\int_0^t \Lambda_s'\Lambda_s \, ds\right) \\ &\quad \prod_{u=1}^{N_t} \exp\left(\frac{(\mu_j^P)^2 - (\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2} J_u^{s,P} + \frac{\mu_v^Q - \mu_v^P}{\mu_v^P \mu_v^Q} J_u^{v,P}\right). \end{aligned} \tag{11}$$

Appendix A shows that Equation (11) is a valid state price density. The first exponential function is the usual Girsanov change of measure of the Brownian motions. The remaining part is the change of measure for the jump component in the stock price and variance.

Equation (11) shows that, in the economy described by this model, price and variance jumps carry a risk premium because when a jump occurs the state price density jumps as well. Bad states of the economy, in which marginal utility is high, can be reached when a negative price jump and/or a positive variance jump occur. When the risk neutral mean of the price jump size is lower than the objective mean, i.e., $\mu_j^Q < \mu_j^P$, and a negative price jump occurs ($J^{s,P} < 0$), the state price density jumps up giving high prices to (Arrow–Debreu) securities with positive payoffs in these bad states of the economy, namely when the stock price falls. Similarly, when the risk neutral mean of the variance jump size is larger than the objective mean, i.e., $\mu_v^Q > \mu_v^P$, and a positive variance jump occurs ($J^{v,P} > 0$), the state price density jumps up when these bad states of the economy occur, namely when volatility is high. In our empirical estimates, we do find that $\mu_j^Q < \mu_j^P$ and $\mu_v^Q > \mu_v^P$.

3. Likelihood-Based Estimation Method

Model (6)–(7) is estimated using the general approach in Aït-Sahalia (2002, 2008). The procedure we employ then combines time series information on the S&P500 returns and cross sectional information on the term structures of VS rates in the same spirit as in other derivative pricing contexts, e.g., Chernov and Ghysels (2000) and Pan (2002). Hence, P and Q parameters, including risk premia, are estimated jointly making the inference procedure theoretically sound.

Let $X_t' = [\log(S_t), Y_t']$ denote the state vector, where $Y_t = [v_t, m_t]'$. The spot variance and its stochastic long run mean, collected in Y_t, are not observed and will be extracted from actual VS rates. The procedure for evaluating the likelihood function consists of four steps. First, we extract the unobserved state vector Y_t from a set of benchmark VS rates, assumed to be observed without error. Second, we evaluate the joint likelihood of the stock returns and extracted time series of latent states, using an approximation to the likelihood function. Third, we multiply this joint likelihood by a Jacobian determinant to compute the likelihood of observed data, namely index returns and term structures of VS rates. Finally, for the remaining VS rates assumed to be observed with error, we calculate the likelihood of the observation errors induced by the extracted state variables. The product of the two likelihoods gives the joint likelihood of the term structures of all VS rates and index returns. We then maximize the joint likelihood over the parameter vector to produce the estimator.

The assumption that a set of benchmark VS rates are observed without error is convenient and standard in the term structure literature because makes the filtering of the latent variables Y_t unnecessary; see, e.g., Pearson and Sun (1994), Aït-Sahalia and Kimmel (2010), and Wu (2011). Alternatively, one could assume that all VS rates are observed with errors, which would require filtering of the latent variables Y_t, for example using Markov Chain Monte Carlo methods as in Eraker (2004).

We found empirically that estimation results are quite insensitive to which VS rates are assumed to be observed with and without errors. This is expected because eventually all VS rates are used in the estimation procedure.

3.1. Extracting State Variables from Variance Swap Rates

Model (6)–(7) implies that the VS rates are affine in the unobserved state variables. This feature suggests the following procedure to extract latent states and motivates our likelihood-based approach.

The unobserved part in the state vector, Y_t, is ℓ dimensional, where $\ell = 2$ in Model (6)–(7). As the method can be applied for $\ell \geq 1$, we describe the procedure for a generic ℓ. At each day t, ℓ VS rates are observed without error, with times to maturities $\tau_1, \ldots, \tau_\ell$. The state vector Y_t is exactly identified by the ℓ VS rates, $\mathrm{VS}_{t,t+\tau_1}, \ldots, \mathrm{VS}_{t,t+\tau_\ell}$. These VS rates jointly follow a Markov process and satisfy

$$\begin{bmatrix} \mathrm{VS}_{t,t+\tau_1} \\ \vdots \\ \mathrm{VS}_{t,t+\tau_\ell} \end{bmatrix} = \begin{bmatrix} a(\tau_1;\Theta) \\ \vdots \\ a(\tau_\ell;\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_1;\Theta)' \\ \vdots \\ b(\tau_\ell;\Theta)' \end{bmatrix} Y_t \tag{12}$$

where Θ denotes the model parameters. Rearranging Equation (9) gives $\mathrm{VS}_{t,t+\tau} = a(\tau;\Theta) +$

$b(\tau;\Theta)'[v_t, m_t]'$, where

$$\begin{aligned} a(\tau;\Theta) &= E^Q[J^2]\lambda_0 + (1+\lambda_1 E^Q[J^2])\left((1-\phi_v^Q(\tau)-\phi_m^Q(\tau))\theta_m^Q + \phi_m^Q(\tau)\mu_v^Q\lambda_0/\tilde{k}_v^Q\right) \\ b(\tau;\Theta)' &= (1+\lambda_1 E^Q[J^2])\ [\phi_v^Q(\tau), \quad \phi_m^Q(\tau)k_v^Q/\tilde{k}_v^Q]. \end{aligned}$$

Equation (12) in vector form reads $\mathrm{VS}_{t,\cdot} = a(\Theta) + b(\Theta)Y_t$, with obvious notation. The current value of the unobserved state vector Y_t can easily be found by solving the equation for Y_t, i.e., $Y_t = b(\Theta)^{-1}[\mathrm{VS}_{t,\cdot} - a(\Theta)]$. The affine relation between VS rates and latent variables makes recovering the latter numerically costless, especially compared to recovering latent variables from standard call and put options as, for e.g., in Pan (2002).

3.2. Likelihood of Stock Returns and Variance Swap Rates Observed Without Error

The extracted time series values of the unobserved state vector Y_t at dates $t_0, t_1, \ldots, t_n$ allows to infer the dynamics of the state variables $X_t' = [\log(S_t), Y_t']$ under the objective probability P. Since the relationship between the unobserved state vector Y_t and VS rates is affine, the transition density of VS rates can be derived from the transition density of Y_t by a change of variables and multiplication by a Jacobian determinant which depends, in this setting, on model parameters but not on the state vector.

Let $p_X(x_\Delta|x_0;\Theta)$ denote the transition density of the state vector X_t under the measure P, i.e., the conditional density of $X_{t+\Delta} = x_\Delta$, given $X_t = x_0$. Let $A_t = [\log(S_t), \mathrm{VS}_{t,t+\tau_1}, \ldots, \mathrm{VS}_{t,t+\tau_\ell}]'$ be the vector of observed asset prices and $p_A(a_\Delta|a_0;\Theta)$ the corresponding transition density. Observed asset prices, A_t, are given by an affine transformation of X_t

$$A_t = \begin{bmatrix} \log(S_t) \\ \mathrm{VS}_{t,\cdot} \end{bmatrix} = \begin{bmatrix} \log(S_t) \\ a(\Theta) + b(\Theta)Y_t \end{bmatrix} = \begin{bmatrix} 0 \\ a(\Theta) \end{bmatrix} + \begin{bmatrix} 1 & 0' \\ 0 & b(\Theta) \end{bmatrix} X_t$$

and rewritten in matrix form reads $A_t = \tilde{a}(\Theta) + \tilde{b}(\Theta)X_t$, with obvious notation. The Jacobian term of the transformation from X_t to A_t is therefore

$$\det\left|\frac{\partial A_t}{\partial X_t'}\right| = \det\left|\tilde{b}(\Theta)\right| = \det\left|b(\Theta)\right|.$$

In Model (6)–(7), $\det|b(\Theta)| = |(1+\lambda_1 E^Q[J^2])^2 \left(\phi_v^Q(\tau_1)\phi_m^Q(\tau_2) - \phi_v^Q(\tau_2)\phi_m^Q(\tau_1)\right) k_v^Q/\tilde{k}_v^Q|$. Since $X_t = \tilde{b}(\Theta)^{-1}[A_t - \tilde{a}(\Theta)]$,

$$p_A(A_\Delta|A_0;\Theta) = \det\left|b(\Theta)^{-1}\right|\ p_X(\tilde{b}(\Theta)^{-1}[A_\Delta - \tilde{a}(\Theta)]|\tilde{b}(\Theta)^{-1}[A_0 - \tilde{a}(\Theta)];\Theta). \tag{13}$$

As the vector of asset prices is Markovian, applying Bayes' Rule, the log-likelihood function of the asset price vector A_t sampled at dates $t_0, t_1, \ldots, t_n$ has the simple form

$$l_n(\Theta) = \sum_{i=1}^{n} l_A(A_{t_i}|A_{t_{i-1}};\Theta) \tag{14}$$

where $l_A = \ln p_A$. As usual in likelihood estimation, we discard the unconditional distribution of the first observation since it is asymptotically irrelevant.

In our applications below, models are estimated using daily data, hence the sampling process is deterministic and $t_i - t_{i-1} = \Delta = 1/252$; see Aït-Sahalia and Mykland (2003) for a treatment of maximum likelihood estimation in the case of randomly spaced sampling times.

3.3. Likelihood of Stock Returns and All Variance Swap Rates

From the coefficients $a(\tau;\Theta)$ and $b(\tau;\Theta)$ and the values of the state vector X_t found in the first step, we can calculate the implied values of the VS rates which are assumed to be observed with error and whose time to maturities are denoted by $\tau_{\ell+1}, \ldots, \tau_{\ell+h}$

$$\begin{bmatrix} \mathrm{VS}_{t,t+\tau_{\ell+1}} \\ \vdots \\ \mathrm{VS}_{t,t+\tau_{\ell+h}} \end{bmatrix} = \begin{bmatrix} a(\tau_{\ell+1};\Theta) \\ \vdots \\ a(\tau_{\ell+h};\Theta) \end{bmatrix} + \begin{bmatrix} b(\tau_{\ell+1};\Theta)' \\ \vdots \\ b(\tau_{\ell+h};\Theta)' \end{bmatrix} Y_t.$$

The observation errors, denoted by $\varepsilon(t, \tau_{\ell+i})$, $i = 1, \ldots, h$, are the differences between such model-based implied VS rates and actual VS rates from the data. By assumption, these errors are Gaussian with zero mean and constant variance, independent of the state process and across time, but possibly correlated across maturities.[16] The joint likelihood of the observation errors can be calculated from the h dimensional Gaussian density function. Since the observation errors are independent of the state variable process, the joint likelihood of stock returns and all observed VS rates is simply the product of the likelihood of stock returns and VS rates observed without error, multiplied by the likelihood of the observation errors. Equivalently, the two log-likelihoods can simply be added to obtain the joint log-likelihood of stock returns and all VS rates.

3.4. *Likelihood Approximation*

Since the state vector X is a continuous-time multivariate jump diffusion process, its transition density is unknown. Since jumps are by nature rare events in a model with finite jump activity, it is unlikely that more than one jump occurs on a single day Δ. This observation motivates the following Bayes' approximation of p_X

$$p_X(x_\Delta|x_0) = p_X(x_\Delta|x_0, N_\Delta = 0)\Pr(N_\Delta = 0) + p_X(x_\Delta|x_0, N_\Delta = 1)\Pr(N_\Delta = 1) + o(\Delta)$$

where $\Pr(N_\Delta = j)$ is the probability that j jumps occur at day Δ, omitting the dependence on

[16]The estimated variances of these errors (reported in Table 3) are very tiny and never induced any sizable probability of negative VS rates.

the parameter Θ for brevity. An extension of the method due to Yu (2007) for jump-diffusion models can provide higher order terms if necessary.

In Model (6)–(7), the largest contribution to the transition density of X (hence to the likelihood) comes from the conditional density that no jump occurs at day Δ. The reason is that the probability of such an event, $\Pr(N_\Delta = 0)$, is typically large and of the order $1-(\lambda_0+\lambda_1 v_0)\,\Delta$. The contribution of the second term is only of the order $(\lambda_0 + \lambda_1 v_0)\,\Delta$. As Δ is one day in our setting, the contribution of higher order terms appears to be quite modest. The main advantage of this approximation is that the leading term, $p_X(x_\Delta|x_0, N_\Delta = 0)$, can be accurately computed using the likelihood expansion method. The expansion for the transition density of X conditioning on no jump has the form of a Taylor series in Δ at order K, with each coefficient $C^{(k)}$ in a Taylor series in $(x - x_0)$ at order $j_k = 2(K - k)$. Denoting $C^{(j_k,k)}$ such expansions, the transition density expansion is

$$\tilde{p}^{(K)}(x|x_0;\Theta) = \Delta^{-(\ell+1)/2} \exp\left[-\frac{C^{(j_{-1},-1)}(x|x_0;\Theta)}{\Delta}\right] \sum_{k=0}^{K} C^{(j_k,k)}(x|x_0;\theta)\frac{\Delta^k}{k!}. \tag{15}$$

Coefficients $C^{(j_k,k)}$ are computed by forcing the Equation (15) to satisfy, to order Δ^K, the forward and backward Kolmogorov equations. A key feature of the method is that the coefficients are obtained in closed form by solving a system of linear equations. This holds true for arbitrary specifications of the dynamics of the state vector X. Moreover, the coefficients need to be computed only once and not at each iteration of the likelihood search. Equation (15) provides a very accurate approximation of the transition density of X already when $K = 2$; e.g., Jensen and Poulsen (2002). In our empirical application below, we use expansions at order $K = 2$.

4. Fitting Variance Swap Rates

4.1. *In-Sample Estimation*

Table 3 reports parameter estimates for Model (6)–(7), based on the in-sample period January

4, 1996 to April 2, 2007. The spot variance is relatively fast mean reverting as k_v^P implies a half-life[17] of 33 days. Its stochastic long run mean is slowly mean reverting with a half-life of about 1.5 years. The instantaneous volatility of v_t is about twice that of m_t. The correlation between stock returns and variance changes, ρ, is -69%, confirming the so-called leverage effect. The long-run average volatility, $\sqrt{\theta_m^P}$, is 20%, in line with the summary statistics in Table 1. Both γ_2 and γ_3 are negative, implying negative instantaneous variance risk premia. The correlation parameter for the VS pricing errors, ρ_e, is slightly negative suggesting that the model does not produce any systematic pricing error.[18]

The expected jump size is negative under the objective probability measure, μ_j^P, and more negative under the risk neutral measure, μ_j^Q, which induces a positive price jump risk premium. The estimate of the jump intensity indicates 2.5 jumps per year on average (i.e., $\lambda_0 + \lambda_1(k_v^Q \theta_m^P + \mu_v^P \lambda_0)/(\kappa_v^P - \mu_v^P \lambda_1))$, which is in line with previous estimates reported in the literature.

Table 3 also reports estimates of three nested models: (i) a two-factor model with price jumps only (labeled SV2F-PJ) with $\mu_v^P = \mu_v^Q = 0$, (ii) a two-factor model with no jump component (labeled SV2F) with the additional restriction $\lambda_0 = \lambda_1 = 0$, and (iii) the Heston model (labeled SV1F) with the additional restriction $m_t = \theta_m^P$ for all t. Imposing each additional restriction significantly deteriorates the fitting of VS rates and S&P500 returns, according to likelihood ratio tests. Thus, Model (6)–(7) outperforms all nested models.

4.2. Out-of-Sample Robustness Checks

We conduct all subsequent analyses using two subsamples. Data from January 4, 1996 to April 2, 2007 are used for in-sample analysis, as Model (6)–(7) is estimated using these data. The remaining sample data, from April 3, 2007 to September 2, 2010, which include the 2007–2009 financial crisis, are used for out-of-sample analysis and robustness checks.

[17]The half-life is defined as the time necessary to halve a unit shock and is given by $-\log(0.5)/k_v^P \times 252$ in number of days.

[18]The determinant of the 3×3 error term correlation matrix is $2\rho_e^3 - 3\rho_e^2 + 1$, which is strictly positive when $\rho_e > -0.5$.

Table 4 shows the pricing errors of Model (6)–(7) when fitting VS rates, for the in- and out-of-sample periods. Pricing errors of the Heston model are also reported for comparison.[19] The pricing error is defined as the model-based VS rate minus the observed VS rate. Model (6)–(7) fits VS rates well both in- and out-of-sample and significantly outperforms the Heston model. For example, its root mean square error is 6 times smaller than that of the Heston model when fitting 24-month to maturity VS rates. The small pricing errors imply that Model (6)–(7) captures the empirical features of VS rates well.

Below, we explore the ability of the model, fitted in-sample, to explain the in-sample realized risk premia and predict the out-of-sample risk premia.

5. Risk Premia: Equity Premium and Volatility Premium

One advantage of modeling the underlying asset returns jointly with the VS rates is that the resulting model produces estimates of risk premia for both sets of variables, including in particular estimates of the classical equity premium. We distinguish between the spot or instantaneous risk premia at each instant t and the integrated ones, defined over each horizon τ.

What have we learned about risk premia that we did not know before? The term structure of integrated equity and variance risk premia, which is largely unexplored in the finance literature, exhibits significant time variation throughout our sample period and large swings during crisis periods. Integrated equity risk premia are countercyclical but the slope of the term structure is procyclical. This indicates that after a market drop investors demand a large risk premium to hold risky stocks, but the risk premium largely depends and strongly decreases with the holding horizon. Integrated variance risk premia become more negative as the horizon increases, especially during turbulent times. This means that, to hedge volatility risk, investors are ready to pay large premia (VS rates are high) and to take large expected losses (variance risk premia are negative and large). Market crashes impact and propagate differently throughout the term

[19] Pricing errors of the two other models in Table 3 are in most cases somewhere in between the pricing errors of the Heston model and Model (6)–(7), and are not reported.

structure of equity and variance risk premia, with the short-end being more affected, and the long-end exhibiting more persistency. Finally, the two term structures respond quite differently to various economic indicators, such as credit spreads, VIX index, and slopes of the interest rate term structure.

5.1. Spot Risk Premia

Model (6)–(7) features four main instantaneous or spot risk premia: A Diffusive Risk Premium (DRP), a Jump Risk Premium (JRP), a Variance Risk Premium (VRP), and a Long-run Mean Risk Premium (LRMRP) which are defined as

$$\mathrm{DRP}_t = (\gamma_1(1-\rho^2)+\gamma_2\rho)v_t, \qquad \mathrm{JRP}_t = (g^P - g^Q)(\lambda_0+\lambda_1 v_t)$$
$$\mathrm{VRP}_t = \gamma_2\sigma_v v_t, \qquad \mathrm{LRMRP}_t = \gamma_3\sigma_m m_t.$$

DRP is the remuneration for diffusive-type risk only (due to the Brownian motions driving the stock price). JRP is the remuneration for the jump component in stock price. The instantaneous Equity Risk Premium (ERP) is the sum of the two, i.e., $\mathrm{ERP}_t = \mathrm{DRP}_t + \mathrm{JRP}_t$.

The mean growth rates of v_t and m_t are different under the probability measures P and Q, and such differences are given by VRP_t and LRMRP_t, respectively. As γ_2 and γ_3 are estimated to be negative (Table 3), VRP and LRMRP are both negative, and on average v_t and m_t are higher under Q than under P. The negative sign of the variance risk premium is not abnormal. The risk premium for return risk is positive, because investors require a higher rate of return as compensation for return risk. On the other hand, investors require a lower level of variance as compensation for variance risk, hence the negative variance risk premium. Risk-averse investors dislike both higher return variance, and higher variance of the return variance.

Table 5 reports the estimated risk premia. During our in-sample period, January 1996 to April 2007, the average ERP is 7%. Notably, approximately 1/3 of the ERP is due to the jump risk premium, which thus accounts for a large fraction of the equity risk premium. Jump prices are rare events (2.5 jumps per year on average), but arguably jump risk is important as it cannot be hedged with any finite number of securities. The average VRP is also substantial and around -8%, while the LRMRP is much lower and around -0.8%. During the out-of-sample period, April 2007 to September 2010, all risk premia almost doubled reflecting the unprecedented turmoil in financial markets around the Lehman Brothers' bankruptcy.

Unreported results show that VRP estimated using the Heston model is only -1.7%, but it increases to -4% for all other nested models with reacher variance dynamics. Heston and two-factor models without jump component imply an ERP of 7%, which is roughly the sum of the DRP and JRP based on Model (6)–(7). This suggests that, in nested models without jump component, all ERP is artificially absorbed by DRP.

As mentioned in Section 2.4, Model (6)–(7) also features a variance jump risk premium, $(\mu_v^P - \mu_v^Q)(\lambda_0 + \lambda_1 v_t)$, which is estimated to be negative but small, as estimates of μ_v^P and μ_v^Q are rather close, and hence it is not reported. This means that setting $\mu_v^P = \mu_v^Q$ as, e.g., in Eraker et al. (2003) and Eraker (2004), does not materially change estimates of risk premia based on Model (6)–(7).

5.2. Integrated Risk Premia

5.2.1 Integrated Equity Risk Premium

We define the annualized integrated Equity Risk Premium (IERP) as

$$\text{IERP}_{t,t+\tau} = E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau \tag{16}$$

which is the ex-ante expected (or forward looking) excess return from buying and holding the S&P500 index from t to $t+\tau$.[20] Extensive research has been devoted to study levels and dynamics of the IERP for a single maturity (often one year, using ex-post measures of the IERP), in particular investigating the so-called equity premium puzzle. Surprisingly, much less

[20]The IERP is the familiar equity risk premium. We use the wording "integrated" to distinguish it from the instantaneous equity risk premium discussed in the previous section.

attention has been devoted to study the term structure of the IERP, which is the focus of this section.

The IERP can be decomposed in the continuous and jump part, i.e., $\text{IERP}_{t,t+\tau} = \text{IERP}^c_{t,t+\tau} + \text{IERP}^j_{t,t+\tau}$, where the continuous part $\text{IERP}^c_{t,t+\tau}$ is the IERP when the jump component is absent, i.e., the jump intensity $\lambda_t = 0$ in the drift μ_t of Model (6), and the jump part $\text{IERP}^j_{t,t+\tau} = \text{IERP}_{t,t+\tau} - \text{IERP}^c_{t,t+\tau}$. This decomposition allows us to quantify how the various risks contribute to the IERP and the corresponding term structure of risk premia.

An advantage of studying the term structure of IERP in a parametric model is that risk premia and their decompositions are exact. Model-free approaches typically involve options, which in turn require interpolations or moving average schemes to reduce the impact on risk premia of time-varying maturities; see Bollerslev and Todorov (2011) for a discussion of this point.

The time-t conditional expectations in (16) can be computed using the transform analysis in Duffie et al. (2000), i.e., solving a system of nonlinear ordinary differential equations derived in Appendix B. The IERP is exponentially affine in the state variables, i.e., $\text{IERP}_{t,t+\tau} = \exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$. Our model estimates in Table 3 imply that $A(\tau)$, $B(\tau)$ and $C(\tau)$ are positive coefficients. Therefore, in quiet times, when the spot variance v_t and its stochastic long run mean m_t are low, IERPs are low as well. When asset prices fall and v_t and/or m_t increase, IERPs increase as well, reflecting distressed asset prices. Thus, the IERP is countercyclical.

To compute the IERP, we use the daily term structure of interest rates, downloaded from OptionMetrics and linearly interpolated to match the VS time to maturities, rather than a constant interest rate as in the analysis above. Table 6 reports mean and standard deviation of the integrated equity risk premium over 2-, 6-, 12- and 24-month horizons.[21] From January 1996 to April 2007, our in-sample period, IERPs are around 6.5% and the term structure is nearly flat.

[21] As the IERP for the 2- and 3-month horizons are rather close, the latter is not reported.

From April 2007 to September 2010, our out-of-sample period, IERPs are significantly larger and about 10%, reflecting distressed asset prices around the Lehman Brothers' bankruptcy. In this period, the term structure of IERPs is downward sloping on average.

Figure 3 shows the evolution of the IERP over time, along with the S&P500 index. The entire term structure of the IERP exhibits significant variation over time, with the short-end being more volatile than the long-end. When the S&P500 steadily increased, such as in 2005–7, the 2-month IERP dropped at the lowest level, around 4%, during our sample period. The term structure was slightly upward sloping with the 24-month IERP at almost 6%. At the end of 2008 and beginning of 2009, after Lehman Brothers collapsed, the term structure of the IERP became significantly downward sloping with the 2-month IERP reaching the highest values in decades. This implies that at the peak of the crisis, investors required equity risk premia as large as 50% to invest in the S&P500 index over short horizons like 2 months, and required less than half these risk premia for investing over long horizons like 2 years.[22] On November 20, 2008, the annualized 2-month IERP was as high as 54%, and between October and December 2008, was above 30% on various occasions, somehow mirroring the fall of the index. Indeed, from mid-September to mid-November 2008, the S&P500 index dropped from 1,200 to 750, loosing 37% of its value. On March 9, 2009, it reached the lowest historical value in more than a decade, at 677, and then recovered 35% of its value within the next two months. Such large swings in the S&P500 index suggest that the large model-based estimates of the IERP are quite sensible. Recently, Martin (2013) provides a model-free lower bound on the equity premium that is by construction lower than, but closely mimics, the equity risk premia depicted in Figure 3.[23]

Table 6 shows that the jump component, $\mathrm{IERP}^{j}_{t,t+\tau}$, contributes significantly to the IERP

[22]It's now obvious in retrospect that Spring 2009 was a great time to go long equities, on the basis of the large equity premium at that point in time, but note that this is here an ex-ante prediction of the model (in fact, made on the basis of the in-sample data only).

[23]van Binsbergen et al. (2013) provide a related study of the term structure of "equity yields," in analogy to bond yields, extracted from dividends derivatives. The term structure of forward equity yields on the S&P500 exhibits similar dynamics as the term structure of equity risk premia depicted in Figure 3. Lettau and Wachter (2007, 2011) provide related studies on the term structure of equity returns, focusing on value and growth stocks.

and its term structure. For example, during our in-sample period, the one-year IERP is 6.3% and 2.5% is due to jump risk. Using a model-free approach, Bollerslev and Todorov (2011) also find that a large fraction of the equity risk premium, around 5% in their study, is due to (large) jump risk, for a short time horizon τ.[24]

To understand which economic factors may drive the term structure of the IERP we conduct regression analysis. We regress the IERP, for each horizon τ, on variables proxying for equity, option, corporate and Treasury bond market conditions, namely daily S&P500 returns, VIX index, the difference between Moody's BAA and AAA corporate bond yields (CScorp, an indicator of credit riskiness within the corporate sector), the difference between Moody's AAA corporate bond yield and 3-month Treasury securities (CSgov, an indicator of credit spread between corporate and Treasury sectors), the difference between the yields on 2-year and 3-month Treasury securities (TermS, the short term slope of the interest rate term structure), the difference between the yields on 10-year and 2-year Treasury securities (TermL, the long term slope of the interest rate term structure). Figure 4 shows the time series plots of the latter four variables.

Panel A in Table 7 summarizes the regression results. Interestingly, these variables have nearly a monotonic (decreasing or increasing) impact on the term structure of IERP, as measured by the slope coefficients. For example, daily S&P500 returns have progressively less negative impact on the IERP as the horizon increases, with the impact becoming statistically insignificant beyond the 3-month horizon. In other words, a negative S&P500 return does increase the IERP but propagates differently throughout the term structure of IERP, with the short-end being more sensitive than the long-end to the shock. An increase of the VIX index has progressively less positive impact on the IERP as the time horizon increases, but the impact remains statistically and economically significant also for the 2-year horizon. CScorp has a positive and decreasing

[24] Bollerslev and Todorov (2011) rely on intraday S&P500 data and SPX options to study the equity risk premium over a single time horizon τ, with median of 14 days.

impact on the IERP, amplifying the countercyclical variation of the IERP, especially in the short-end of the term structure. This suggests that strains in the corporate sector impact the IERP but only over short horizons of a few months.

The slope coefficients of other variables change sign throughout the term structure of IERP, for example from positive to negative for TermL, the long term slope of the yield curve. During the market drop in Fall 2008, TermL increased (Figure 4). Consequently, the positive slope coefficients for short term IERP and negative slope coefficients for long term IERP amplified the downward slope of the IERP term structure during those turbulent times. To the extent that this variable reflects "flight-to-liquidity" phenomena during the crisis (i.e., investors rebalancing their portfolios from equities to treasuries), VS market participants appear to regard these phenomena as transient and to anticipate an overall portfolio rebalancing (from treasuries to equities) when the crisis will be over.

All in all, economic indicators appear to have a rich impact on equity risk premia over different horizons. This impact can be uncovered only by studying the term structure of the IERP.

5.2.2 Integrated Variance Risk Premium

We define the annualized integrated variance risk premium (IVRP) as $\mathrm{IVRP}_{t,t+\tau} = E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}]$, which represents the ex-ante expected profit to the long side of a VS contract, when the position is entered at time t and held till maturity $t + \tau$. Table 6 reports summary statistics of the IVRP and Figure 5 shows the dynamic over time. The average IVRP for 24-month maturity is -2.9% during our out-of-sample period and can be as high as -5% in variance units. These are large risk premia compared to an average spot variance of 4% in variance units. While Model (6)–(7) is flexible enough to generate positive and negative IVRP, estimated ex-ante IVRP is always negative. This indicates that investors perceive volatility increases as unfavorable events and are willing to take large expected losses to buy protection against such volatility increases.

The longer the time to maturity the higher in absolute value the annualized IVRP. Thus, the term structure of IVRP is on average downward sloping, i.e., long-term VS contracts carry more risk premium for stochastic variance than short-term contracts. In fact, Filipović et al. (2015) show that an optimal investment strategy is to go short in long-term VS (to earn the risk premium) and to go long in short-term VS (to hedge volatility risk).[25]

Similarly to the IERP, we conduct regression analysis to understand which economic factors may drive the term structure of the IVRP. Panel B in Table 7 summarizes the regression results. A negative S&P500 return induces a more negative IVRP, especially for the short-end of the term structure, and the effect becomes statistically insignificant only beyond a six-month horizon. An increase of the VIX index also induces a more negative IVRP but its impact is quite uniform, statistically and economically significant, throughout the IVRP term structure. Thus, despite being a 30-day volatility index, the VIX behaves more like a "level factor" than a short term factor for variance risk premia. CScorp has a negative and decreasing impact on the IVRP, amplifying the procyclical variation of the IVRP. Thus destressed conditions in the corporate sector appear to command a variance risk premium, but mainly over short horizons. The slope coefficients of other variables change sign throughout the IVRP term structure, and thus impact the slope of the term structure. For example the regression coefficients of TermL range from negative to positive as the time horizon increases. An increase of the slope of the yield curve during Fall 2008 tends to induce an upward sloping term structure of the IVRP. However, in contrast to its impact on the IERP term structure, this effect is not very pronounced, and the IVRP term structure remains essentially downward sloping.

As the quadratic variation can be naturally decomposed in the continuous, $\mathrm{QV}^c_{t,t+\tau}$, and

[25]Egloff et al. (2010) also study optimal investment in VS but they reach the opposite conclusion for the optimal allocation. This can be explained by the different stochastic volatility models, investment strategies and market price of risk specifications used in the two studies.

discontinuous, $\mathrm{QV}^j_{t,t+\tau}$, part (see Equation (3)), the IVRP can also be decomposed as

$$\begin{aligned}
\mathrm{IVRP}_{t,t+\tau} &= E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}] \\
&= (E_t^P[\mathrm{QV}^c_{t,t+\tau}] - E_t^Q[\mathrm{QV}^c_{t,t+\tau}]) + (E_t^P[\mathrm{QV}^j_{t,t+\tau}] - E_t^Q[\mathrm{QV}^j_{t,t+\tau}]) \\
&= \mathrm{IVRP}^c_{t,t+\tau} + \mathrm{IVRP}^j_{t,t+\tau}.
\end{aligned}$$

We now investigate the impact of negative price jumps and the induced term structure of variance risk premia. As many investors are "long in the market" and the leverage effect is very pronounced, negative price jumps are perceived as unfavorable events and thus can carry particular risk premia. The contribution of negative price jumps to the IVRP is given by

$$\mathrm{IVRP}(k)^j_{t,t+\tau} = E_t^P[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}] - E_t^Q[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}]$$

where $1\{J^s < k\}$ is the indicator function of the event $J^s < k$. We set $k = -1\%$, i.e., we study the contribution of daily jumps below -1% to the IVRP.[26] Similar values of the threshold k produce similar results for $\mathrm{IVRP}(k)^j_{t,t+\tau}$. Given Model (6)–(7), $\mathrm{IVRP}(k)^j_{t,t+\tau}$ is available in closed form.

Table 6 reports summary statistics for $\mathrm{IVRP}(k)^j_{t,t+\tau}$, when $k = -1\%$. Since $\mathrm{IVRP}(k)^j_{t,t+\tau}$ is essentially constant when the time horizon τ increases, its relative contribution to the IVRP is decreasing on average and thus largest for the 2-month IVRP. In other words, short-term variance risk premia appear to reflect investors' fear of a market drop, rather than the impact of stochastic volatility on the investment opportunity set. Although price jumps below -1% are infrequent events, their contribution to short-term IVRP is substantial. For the 2-month horizon, $\mathrm{IVRP}(k)^j_{t,t+\tau}$ accounts for about 20% of the IVRP.

Figure 5 shows the term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ over time. Similarly to the IVRP, the term

[26]From January 1996 to September 2010, daily S&P500 returns are on average 3 times a month below -1%.

structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ is generally downward sloping in quiet times. However, in contrast to IVRP, during market crashes the term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ becomes suddenly upward sloping, reflecting the large jump risk due to a price fall. As an example, in Fall 2008 the whole term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ moved downward but the two-month $\mathrm{IVRP}(k)^j_{t,t+\tau}$ exhibited the largest negative drop and took several months to revert to average values. The 12- and 24-month $\mathrm{IVRP}(k)^j_{t,t+\tau}$ took even longer to revert to average values. All in all, these findings suggest that investors' willingness to ensure against a market crash increases after a price fall with a persistent impact on the IVRP. The dynamics of the term structure of $\mathrm{IVRP}(k)^j_{t,t+\tau}$ further show that the price fall has the strongest impact on the short-term IVRP but the persistency is more pronounced for long-term IVRP.

In order to examine the extent to which the large variance risk premia potentially translate into economic gains, we consider a simple but relatively robust trading strategy involving VS. The trading strategy is robust in the sense that Model (6)–(7) and corresponding estimates are used only to decide whether or not to invest in VS, i.e., to extract a trading signal.

Since realized variances are lower on average than VS rates, shorting VS contracts generates a positive return on average. Such a trading strategy can be improved as follows. At each day t, we compute the expected profit from shorting a VS contract, i.e., $\mathrm{VS}_{t,t+\tau} - E^P_t[\mathrm{QV}_{t,t+\tau}]$. Then, the strategy is to short the VS contract only when the expected profit is large enough and precisely n times larger than its expected standard deviation. When $n = 0$, the VS contract is shorted as soon as the expected profit is positive. When $n > 0$, the contract is shorted less often. If at day t the VS contract is shorted, we compute the actual return from the investment by comparing the VS rate and the ex-post realized variance, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$. Since the strategy is short-and-hold (conditional on a model-based signal), transaction costs are unlikely to affect the results and will not be considered. If at day t the VS is not shorted, the return from t to $t + \tau$ is obviously zero and not considered when assessing the performance of the strategy. We repeat this procedure for each day t in our sample.

As a benchmark, we consider the following trading strategy based on the S&P500 index. If at any day t the VS contract with maturity $t + \tau$ is shorted, we invest \$1 in the S&P500 index and liquidate the position at day $t + \tau$. Thus, the investment horizon is the same as the one for the VS strategy. The actual return is computed using S&P500 index prices.

Table 8 compares the trading strategies using the classical Sharpe ratios. Given the non-normality of returns, Sharpe ratios need to be cautiously interpreted. We also computed Sortino ratios[27] and results were very similar, and not reported. As the VS is a forward contract, Sharpe ratios of the corresponding short-and-hold strategies are calculated simply as the average return throughout our sample divided by its standard deviation. To compute Sharpe ratios of buy-and-hold strategies with the S&P500 index, we use the daily term structure of interest rates, downloaded from OptionMetrics and linearly interpolated to match the various investment horizons. We experimented other values of interest rates, such as a constant rate of zero or 4%, and the results reported in Table 8 change only marginally.

Shorting VS appears to be significantly more profitable than investing in the S&P500 index, over the same time horizons. This suggests that VS contracts offer economically important investment opportunities. It also confirms our model-based finding that investors are ready to pay high "insurance premia" to obtain protection against volatility increases.

When the threshold n increases, the VS is shorted less often.[28] As shown in Table 8, Sharpe ratios from investing in VS are nearly uniformly and significantly increasing in the threshold n. Thus, Model (6)–(7) seems to provide valuable information to generate a trading signal for shorting variance swaps.

Figure 6 shows the returns of the short-and-hold trading strategy based on 12-month VS

[27] The Sortino ratio is a popular performance measure and defined as the mean return in excess of a minimum acceptable return divided by the downside deviation. This ratio penalizes only returns below the minimum acceptable return, in contrast to the standard deviation that equally penalizes returns below and above the average return. In our computations we set the minimum acceptable return to zero, and the Sortino ratio is $(\sum_{t=1}^{T} r_t/T)/\sigma_D$, where r_t is the time-t return of a given trading strategy, the downside variance $\sigma_D^2 = \sum_{t=1}^{T}(r_t\,1\{r_t < 0\})^2/T$ and T is the total number of returns.

[28] For example, the 12-month VS contract is shorted 80%, 59% and 23% of the times when $n = 1/4$, $1/2$, 1, respectively.

and the long-and-hold trading strategy based on the S&P500 index. With the exception of 2008, shorting VS tends to provide stable and substantial positive returns. The losses during 2008 reflect jump and volatility risk that short positions are carrying, but they are smaller than the losses from the buy-and-hold S&P500 strategy. Long positions in the S&P500 generate substantial more volatile returns. Interestingly, shorting VS does not appear to suffer from the "picking up nickels in front of steamroller" syndrome during the period we looked at, despite the inclusion out-of-sample of the 2007–2009 financial crisis.

Finally, does shorting VS provide any diversification benefit? Table 9 shows correlations between daily returns of short positions in VS, long positions in the S&P500 index, and Treasury bond yields over the same time horizons. Short positions in VS are generally positively correlated with long positions in the S&P500 and, consistently with the patterns of the integrated risk premia, more so during turbulent than quiet times. They are also generally negatively correlated with long bond positions.

5.3. *Risk Premia: Robustness Checks*

To check the robustness of the parametric model, we note that the change of measure in Equation (11) implies that the mean jump size is different, not the jump intensity, under P and Q. Now we let the jump intensity be $\lambda_t^P = \lambda_0^P + \lambda_1^P v_t$ under P and $\lambda_t^Q = \lambda_0^Q + \lambda_1^Q v_t$ under Q. The drift under P of the index price process becomes

$$\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + g^P(\lambda_0^P + \lambda_1^P v_t) - g^Q(\lambda_0^Q + \lambda_1^Q v_t)$$

and jump risk premia become

$$\begin{aligned}
\mathrm{JRP}_t &= g^P(\lambda_0^P + \lambda_1^P v_t) - g^Q(\lambda_0^Q + \lambda_1^Q v_t) \\
\mathrm{IVRP}^j_{t,t+\tau} &= E^P[(J^s)^2](\lambda_0^P + \lambda_1^P E_t^P[\mathrm{QV}^c_{t,t+\tau}]) - E^Q[(J^s)^2](\lambda_0^Q + \lambda_1^Q E_t^Q[\mathrm{QV}^c_{t,t+\tau}]).
\end{aligned}$$

Estimation results of this more general model imply nearly the same dynamics for spot variance, stochastic long run mean, instantaneous risk premia, and integrated risk premia due to the continuous part of the quadratic variation. However, the estimated overall risk neutral jump-intensity, λ_t^Q, turns out to be smaller than objective jump-intensity, λ_t^P. Pan (2002) reports the same finding using her stochastic volatility model.[29]

6. Conclusion

We study the term structure of variance swaps, equity and variance risk premia. Comparing VIX-type indices from the option market and VS rates, we find evidence for a large price jump component in VS rates. This suggests that either the jump risk is heavily priced by VS traders or some segmentation between the VS and option markets exists or both.

Based on our model estimates, the term structure of variance risk premia appears to be negative and downward sloping, and the short-end of the term structure mainly reflects investors' fear of a market drop, rather than the impact of stochastic volatility on the investment set. Moreover, investors' willingness to ensure against volatility risk appears to increase after a market crash. This effect is stronger for short horizons and more persistent for long horizons.

We find that the term structure of equity risk premia, i.e., the expected excess returns from buying and holding the S&P500 over fixed horizons, is countercyclical while its slope is procyclical. Thus, during crisis periods investors demand large risk premia for holding equities, but the risk premia largely depend and strongly decrease with the holding horizon. Finally, economic indicators proxying for equity, option, corporate and Treasury bond market conditions appear to have a rich and different impact throughout the term structure of equity and variance risk premia.

[29]Pan considers jump intensities $\lambda_1^P v_t$ under P and $\lambda_1^Q v_t$ under Q, in our notation, and defines the jump-timing risk premium as $\lambda_1^Q - \lambda_1^P$, the opposite of our definition. Note that Pan's specification of jump intensities can be recovered setting $\lambda_0^P = \lambda_0^Q = 0$ in our model.

A. Pricing Kernel

Recall that the market price of risks for the Brownian motions are

$$\Lambda_t' = [\gamma_1\sqrt{(1-\rho^2)v_t}, \quad \gamma_2\sqrt{v_t}, \quad \gamma_3\sqrt{m_t}].$$

We define the pricing kernel (or Stochastic Discount Factor) as

$$\pi_t = e^{-rt}\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-rt - \int_0^t \Lambda_u'\,dW_u^P - \frac{1}{2}\int_0^t \Lambda_u'\Lambda_u\,du\right)\prod_{u=1}^{N_t}\exp\left(a_j + b_j J_u^{s,P} + c_j J_u^{v,P}\right)\frac{\mu_v^P}{\mu_v^Q}$$

where $a_j = ((\mu_j^P)^2 - (\mu_j^Q)^2)/(2\sigma_j^2)$, $b_j = (\mu_j^Q - \mu_j^P)/\sigma_j^2$, and $c_j = (\mu_v^Q - \mu_v^P)/(\mu_v^P\mu_v^Q)$. The process π_t is a valid pricing kernel when deflated bank account and deflated cum-dividend price processes are P-martingales.

When a jump occurs the pricing kernel jumps from $\pi_{t\_}$ to $\pi_t = \pi_{t\_}e^{a_j + b_j J_t^{s,P} + c_j J_t^{v,P}}\frac{\mu_v^P}{\mu_v^Q}$, hence

$$\begin{aligned}
\frac{d\pi_t}{\pi_t} &= -r\,dt - \Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P \qquad (17)\\
&= -r\,dt - (\gamma_1\sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \gamma_2\sqrt{v_t}\,dW_{2t}^P + \gamma_3\sqrt{m_t}\,dW_{3t}^P)\\
&\quad +(\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P.
\end{aligned}$$

The typical expression for the dynamic of π_t includes an explicit compensator for the jump term to emphasize that $E_t^P[d\pi_t/\pi_t] = -r\,dt$. Here we use a different expression for the jump term in π_t to obtain an interpretation for the coefficients a_j, b_j and c_j. As we show below, the jump term in (17) is already compensated.

Let $B_t = e^{rt}$ denote the bank account level and $B_t^\pi = B_t\pi_t$ the deflated bank account. Applying Itô's formula

$$\begin{aligned}
d(B_t^\pi) &= B_t\,d\pi_t + \pi_t\,dB_t\\
&= B_t^\pi(-r\,dt - \Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P) + B_t^\pi r\,dt\\
d(B_t^\pi)/B_t^\pi &= -\Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P.
\end{aligned}$$

Hence, B_t^π is a P-martingale (or has zero drift) when $E^P[\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = 1$. As $J^{s,P}$ and $J^{v,P}$ are independent, the last equation holds when $E^P[\exp(a_j + b_j J_t^{s,P})] = 1$ and $E^P[\exp(c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = 1$, which is shown in the following calculations:

$$\begin{aligned}
E^P[\exp(a_j + b_j J_t^{s,P})] &= \exp(a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2})\\
a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2} &= \frac{(\mu_j^P)^2 - (\mu_j^Q)^2}{2\sigma_j^2} + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\mu_j^P + \left(\frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\right)^2\frac{\sigma_j^2}{2}\\
&= \frac{(\mu_j^P)^2 - (\mu_j^Q)^2 + 2\mu_j^Q\mu_j^P - 2(\mu_j^P)^2 + (\mu_j^Q)^2 + (\mu_j^P)^2 - 2\mu_j^Q\mu_j^P}{2\sigma_j^2} = 0
\end{aligned}$$

where we used $J^{s,P} \sim \mathcal{N}(\mu_j^P, \sigma_j^2)$. As $J^{v,P} \sim Exp(\mu_v^P)$

$$E^P[\exp(c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = \frac{\mu_v^P}{\mu_v^Q}\int_0^\infty e^{c_j J^v}\frac{e^{-J^v/\mu_v^P}}{\mu_v^P}\,dJ^v = 1.$$

Let $S_{\delta,t} = S_t e^{\delta t}$ denote the cum-dividend stock price, hence

$$\begin{aligned}\frac{dS_{\delta,t}}{S_{\delta,t}} = \frac{dS_t}{S_t} + \delta\,dt &= (r + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \rho\sqrt{v_t}\,dW_{2t}^P \\ &\quad +(\exp(J_t^{s,P}) - 1)\,dN_t^P\end{aligned}$$

where S_t is the ex-dividend stock price. Let $S_{\delta,t}^\pi$ be the deflated cum-dividend stock price, i.e., $S_{\delta,t}\,\pi_t$. When a jump occurs, both π_t and S_t jump and S_δ^π jumps from S_{δ,t_-}^π to $S_{\delta,t}^\pi = S_{\delta,t_-}^\pi \exp(a_j + b_j J_t^{s,P} + J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}$. Hence, at the jump time, $dS_{\delta,t}^\pi / S_{\delta,t}^\pi = \exp(a_j + (b_j + 1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1$.

Applying Itô's formula, with π_t^c and $S_{\delta,t}^c$ denoting the continuous part of π_t and $S_{\delta,t}$, respectively,

$$\begin{aligned}dS_{\delta,t}^\pi &= S_{\delta,t}\,d\pi_t^c + \pi_t\,dS_{\delta,t}^c + dS_{\delta,t}^c\,d\pi_t^c + S_{\delta,t}\pi_t(\exp(a_j + (b_j+1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P \\ &= S_{\delta,t}\,\pi_t(-r\,dt - \gamma_1\sqrt{(1-\rho^2)v_t}\,dW_{1t}^P - \gamma_2\sqrt{v_t}\,dW_{2t}^P - \gamma_3\sqrt{m_t}\,dW_{3t}^P) \\ &\quad +\pi_t\,S_{\delta,t}((r + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t - g^Q\lambda_t)\,dt + \sqrt{(1-\rho^2)v_t}\,dW_{1t}^P + \rho\sqrt{v_t}\,dW_{2t}^P) \\ &\quad -S_{\delta,t}\pi_t(\gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t)\,dt + S_{\delta,t}\pi_t(\exp(a_j + (b_j+1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P \\ \frac{dS_{\delta,t}^\pi}{S_{\delta,t}^\pi} &= \sqrt{(1-\rho^2)v_t}(1-\gamma_1)\,dW_{1t}^P + (\rho - \gamma_2)\sqrt{v_t}\,dW_{2t}^P - \gamma_3\sqrt{m_t}\,dW_{3t}^P \\ &\quad +(\exp(a_j + (b_j+1)J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P - g^Q\lambda_t\,dt.\end{aligned}$$

Hence, $S_{\delta,t}^\pi$ is a P-martingale (or has zero drift) when $E^P[\exp(c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q}] = 1$, which we already showed above, and when $E^P[\exp(a_j + (b_j+1)J_t^{s,P}) - 1] = g^Q$, which is indeed the case as shown in the following calculations:

$$\begin{aligned}E^P[\exp(a_j + (b_j+1)J_t^{s,P}) - 1] &= g^Q \\ \exp(a_j + (b_j+1)\mu_j^P + (b_j+1)^2\frac{\sigma_j^2}{2}) - 1 &= \exp(\mu_j^Q + \frac{\sigma_j^2}{2}) - 1 \\ a_j + b_j\mu_j^P + \mu_j^P + b_j^2\frac{\sigma_j^2}{2} + 2b_j\frac{\sigma_j^2}{2} &= \mu_j^Q \\ \mu_j^P + \frac{\mu_j^Q - \mu_j^P}{\sigma_j^2}\sigma_j^2 &= \mu_j^Q\end{aligned}$$

where we used $a_j + b_j\mu_j^P + b_j^2\frac{\sigma_j^2}{2} = 0$, which is implied by the martingale property of the deflated bank account.

Finally, the relation between the pricing kernel π_t and the risk-neutral dynamics is derived as usual. Define the density process $\xi_t = \pi_t e^{rt}$. Under usual technical conditions, applying Itô's formula, $d\xi_t/\xi_t = -\Lambda_t'\,dW_t^P + (\exp(a_j + b_j J_t^{s,P} + c_j J_t^{v,P})\frac{\mu_v^P}{\mu_v^Q} - 1)\,dN_t^P$, which shows that ξ_t is a P-martingale and hence it uniquely defines an equivalent martingale measure Q. Defining the Q-Brownian motions as $dW_{1t}^Q = dW_{1t}^P + \gamma_1\sqrt{(1-\rho^2)v_t}\,dt$, $dW_{2t}^Q = dW_{2t}^P + \gamma_2\sqrt{v_t}\,dt$ and $dW_{3t}^Q = dW_{3t}^P + \gamma_3\sqrt{m_t}\,dt$, gives the risk-neutral dynamic of the stock price S, spot variance v, and stochastic long run mean m in Equation (7).

B. Integrated Equity Risk Premia

To compute the IERP in (16) we rely on the transform analysis of Duffie et al. (2000), which is often used in finance applications; e.g., Duffie et al. (2003). In this appendix we provide a self-contained application of this theory to the calculation of the IERP in our setting.

The basic step is to compute a conditional expectation of the form $E_t^P[\exp(\zeta\int_t^{t+\tau} v_s\,ds)]$, where ζ is a given constant. The first conditional expectation in (16) is $E_t^P[S_{t+\tau}/S_t] = E_t^P[\exp(\int_t^{t+\tau}\mu_s\,ds)]$, where μ_s is an affine function of v_s, defined after (6).[30]

Define the stochastic process $\psi_t = E_t^P[\exp(\zeta\int_0^T v_s\,ds)]$, which is a P-martingale by construction for all $t \geq 0$, under standard integrability conditions. Guess the functional form $\psi_t = \exp(\zeta\int_0^t v_s\,ds)\exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$, which is exponentially affine in the state variables v_t and m_t. Recall $\tau = T - t$. The necessary derivatives to apply Itô's formula to ψ_t are

$$\begin{aligned}
\frac{\partial\psi_t}{\partial t} &= \psi_t(\zeta v_t - A(\tau)' - B(\tau)'v_t - C(\tau)'m_t) \\
\frac{\partial\psi_t}{\partial v_t} &= \psi_t B(\tau), \qquad \frac{\partial^2\psi_t}{\partial v_t^2} = \psi_t B(\tau)^2 \\
\frac{\partial\psi_t}{\partial m_t} &= \psi_t C(\tau), \qquad \frac{\partial^2\psi_t}{\partial m_t^2} = \psi_t C(\tau)^2
\end{aligned}$$

If a jump occurs at time t, the spot variance jumps from v_{t_-} to $v_t = v_{t_-} + J_t^{v,P}$, and consequently the process ψ jumps from ψ_{t_-} to ψ_t, which implies that

$$\frac{\psi_t}{\psi_{t_-}} - 1 = \frac{e^{\zeta\int_0^t v_s\,ds}\;e^{A(\tau)+B(\tau)v_t}}{e^{\zeta\int_0^t v_s\,ds}\;e^{A(\tau)+B(\tau)v_{t_-}}} - 1 = e^{B(\tau)(v_t - v_{t_-})} - 1 = e^{B(\tau)J_t^{v,P}} - 1.$$

Rewriting the P-dynamic of the spot variance, with obvious notation, as

$$dv_t = (k_v^Q m_t - k_v^P v_t)\,dt + \sigma_v\sqrt{v_t}\,dW_{2t}^P + J_t^{v,P}\,dN_t = dv_t^{cont} + J_t^{v,P}\,dN_t$$

[30]The second conditional expectation is simply $E_t^Q[S_{t+\tau}/S_t] = \exp((r_{t,t+\tau} - \delta)\tau)$, assuming a time varying but deterministic term structure of interest rates.

and applying Itô's formula to ψ_t gives

$$
\begin{aligned}
\frac{d\psi_t}{\psi_{t_-}} &= (\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t)\,dt + B(\tau)(dv_t^{cont}) + \frac{1}{2}B(\tau)^2(dv_t^{cont})^2 \\
&\quad + C(\tau)(dm_t) + \frac{1}{2}C(\tau)^2(dm_t)^2 + (\frac{\psi_t}{\psi_{t_-}} - 1)dN_t \\
&= (\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t)\,dt + B(\tau)(k_v^Q m_t - k_v^P v_t)dt + \sigma_v\sqrt{v_t}dW_{2t}^P) + \frac{1}{2}B(\tau)^2\sigma_v^2 v_t\,dt \\
&\quad + C(\tau)(k_m^P(\theta_m^P - m_t)dt + \sigma_m\sqrt{m_t}dW_{3t}^P) + \frac{1}{2}C(\tau)^2\sigma_m^2 m_t\,dt \\
&\quad + (e^{B(\tau)J_t^{v,P}} - 1)dN_t - E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\,dt + E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\,dt \\
&= (\zeta v_t - A(\tau)' - B(\tau)' v_t - C(\tau)' m_t)\,dt + B(\tau)(k_v^Q m_t - k_v^P v_t)dt + \frac{1}{2}B(\tau)^2\sigma_v^2 v_t\,dt \\
&\quad + C(\tau)k_m^P(\theta_m^P - m_t)dt + \frac{1}{2}C(\tau)^2\sigma_m^2 m_t\,dt \\
&\quad + E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\,dt + dM_t^P
\end{aligned}
$$

where $dM_t^P = \sigma_v\sqrt{v_t}\,dW_{2t}^P + \sigma_m\sqrt{m_t}\,dW_{3t}^P + (e^{B(\tau)J_t^{v,P}} - 1)dN_t - E^P[e^{B(\tau)J_t^v} - 1](\lambda_0 + \lambda_1 v_t)\,dt$ is a P-martingale increment.

As ψ_t is a P-martingale, the drift must be zero for each time t and level of the state variables v_t and m_t. Collecting terms in dt, $v_t\,dt$ and $m_t\,dt$, respectively, and setting them equal to zero, give three nonlinear ordinary differential equations

$$
\begin{aligned}
0 &= -A(\tau)' + C(\tau)k_m^P\theta_m^P + E^P[e^{B(\tau)J_t^v} - 1]\lambda_0 \\
0 &= \zeta - B(\tau)' - B(\tau)k_v^P + \frac{1}{2}B(\tau)^2\sigma_v^2 + E^P[e^{B(\tau)J_t^v} - 1]\lambda_1 \\
0 &= -C(\tau)' + B(\tau)k_v^Q - C(\tau)k_m^P + \frac{1}{2}C(\tau)^2\sigma_m^2
\end{aligned}
$$

for the coefficients $A(\tau)$, $B(\tau)$ and $C(\tau)$, with terminal conditions $A(0) = B(0) = C(0) = 0$. As the system is time-homogenous, for each time horizon τ, these coefficients need to be computed only once. Thus, at each time t, $E_t^P[\exp(\zeta\int_t^{t+\tau} v_s\,ds)] = \exp(A(\tau) + B(\tau)v_t + C(\tau)m_t)$.

The expectation in the first two differential equations is

$$
E^P[e^{B(\tau)J_t^v}] = \int_0^\infty e^{B(\tau)J^v}\frac{e^{-J^v/\mu_v^P}}{\mu_v^P}\,dJ^v = \frac{1}{\mu_v^P}\int_0^\infty e^{-J^v\left(\frac{1}{\mu_v^P} - B(\tau)\right)}\,dJ^v = \frac{1}{1 - B(\tau)\mu_v^P}
$$

and the integral above converges when $\left(\frac{1}{\mu_v^P} - B(\tau)\right) > 0$, which is indeed the case according to our estimates. Then

$$
E^P[e^{B(\tau)J_t^v} - 1] = \frac{B(\tau)\mu_v^P}{1 - B(\tau)\mu_v^P}
$$

is substituted in the first two differential equations, and the system is solved numerically.

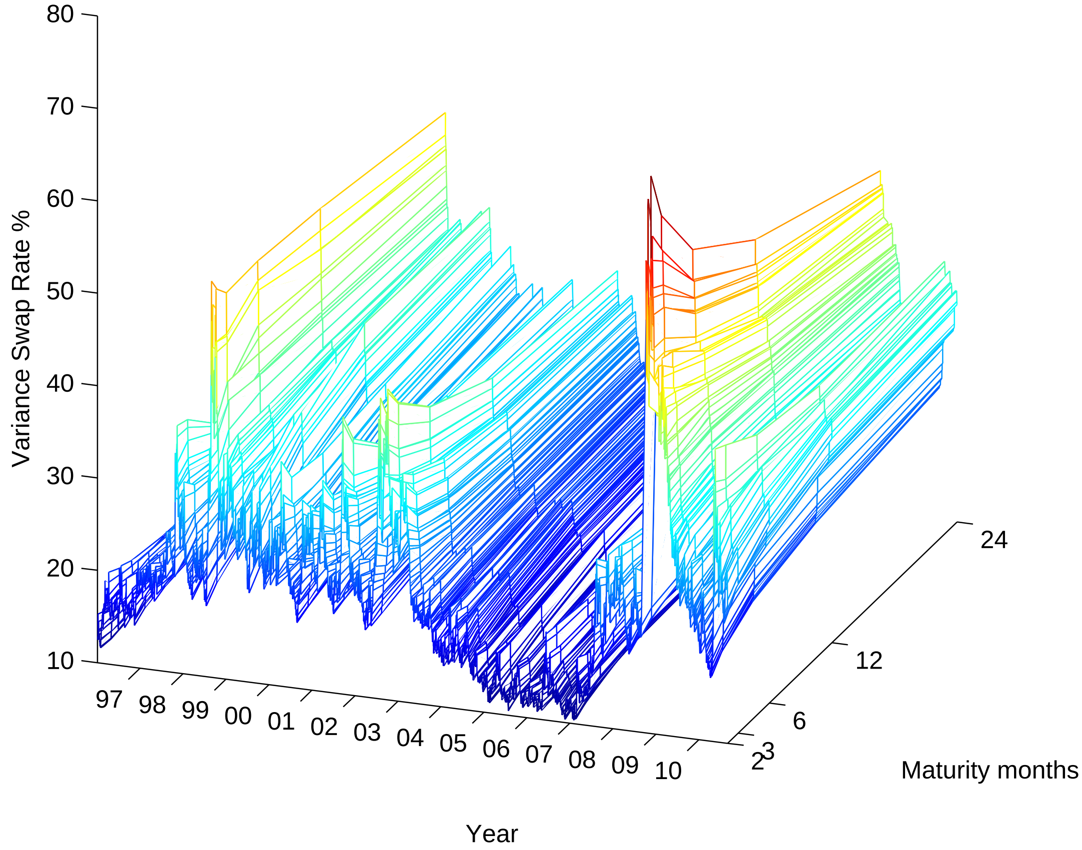


Figure 1. Term structure of variance swap rates. Values are in volatility percentage units, i.e., $\mathrm{VS}_{t,t+\tau}^{1/2} \times 100$, with 2-, 3-, 6-, 12-, and 24-month to maturity from January 4, 1996 to September 2, 2010, that are 3,624 observations for each time to maturity.

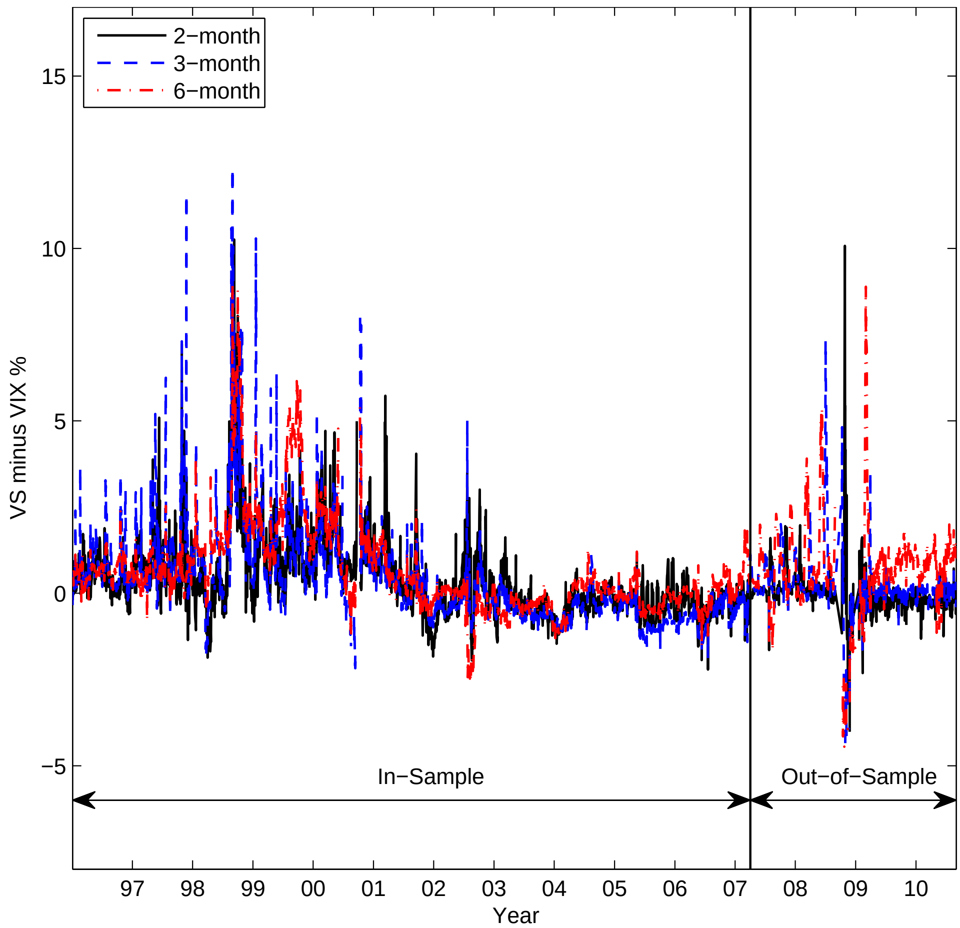


Figure 2. Term structure of model-free jump component in variance swap rates. VS rates minus calculated VIX-type indices for 2-, 3-, and 6-month to maturity from January 4, 1996 to September 2, 2010, that are 3,624 observations for each maturity. The difference is in volatility percentage units, i.e., $(\mathrm{VS}_{t,t+\tau}^{1/2} - \mathrm{VIX}_{t,t+\tau}^{1/2}) \times 100$.

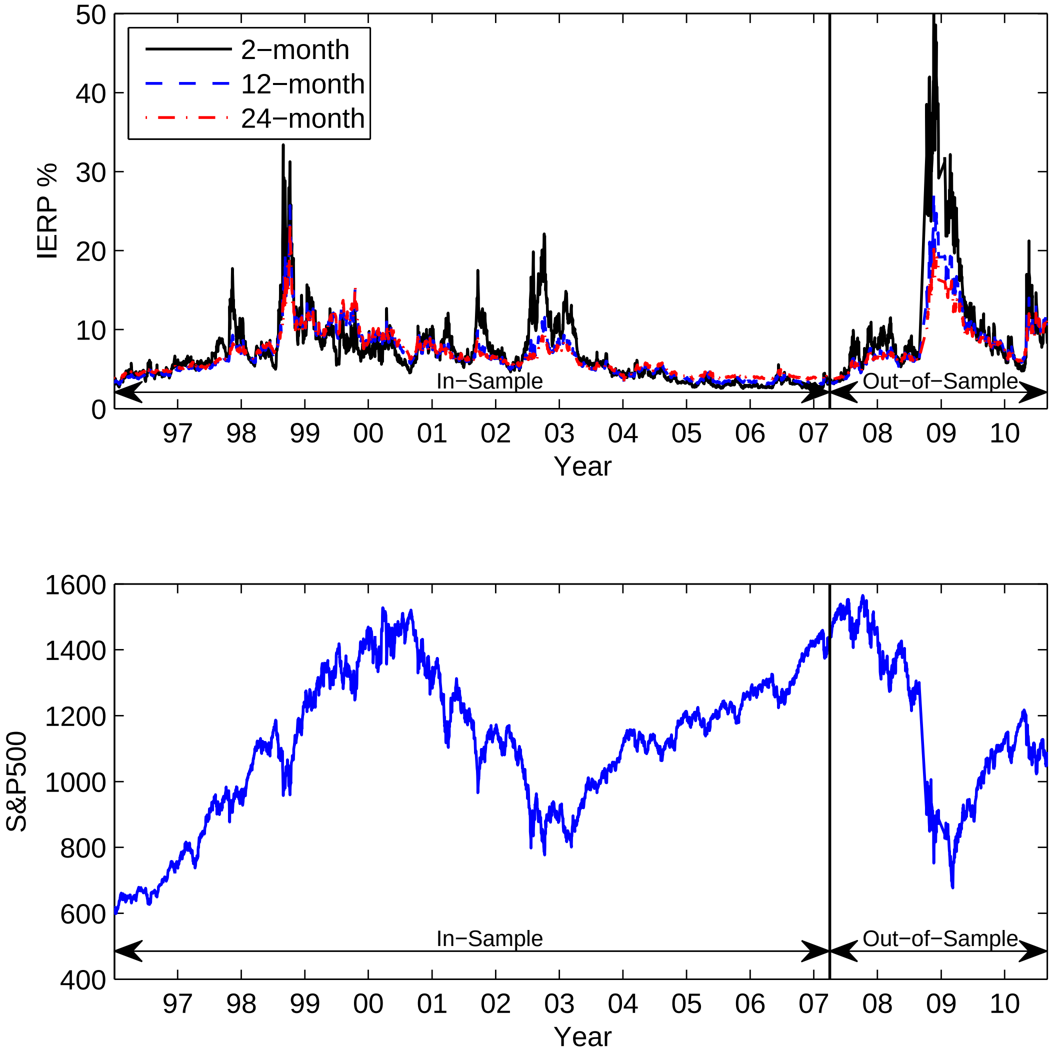


Figure 3. Term structure of integrated equity risk premia and S&P500 index. Upper graph: annualized integrated equity risk premia, i.e., $(E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau) \times 100$. Lower graph: S&P500 index, S_t. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.

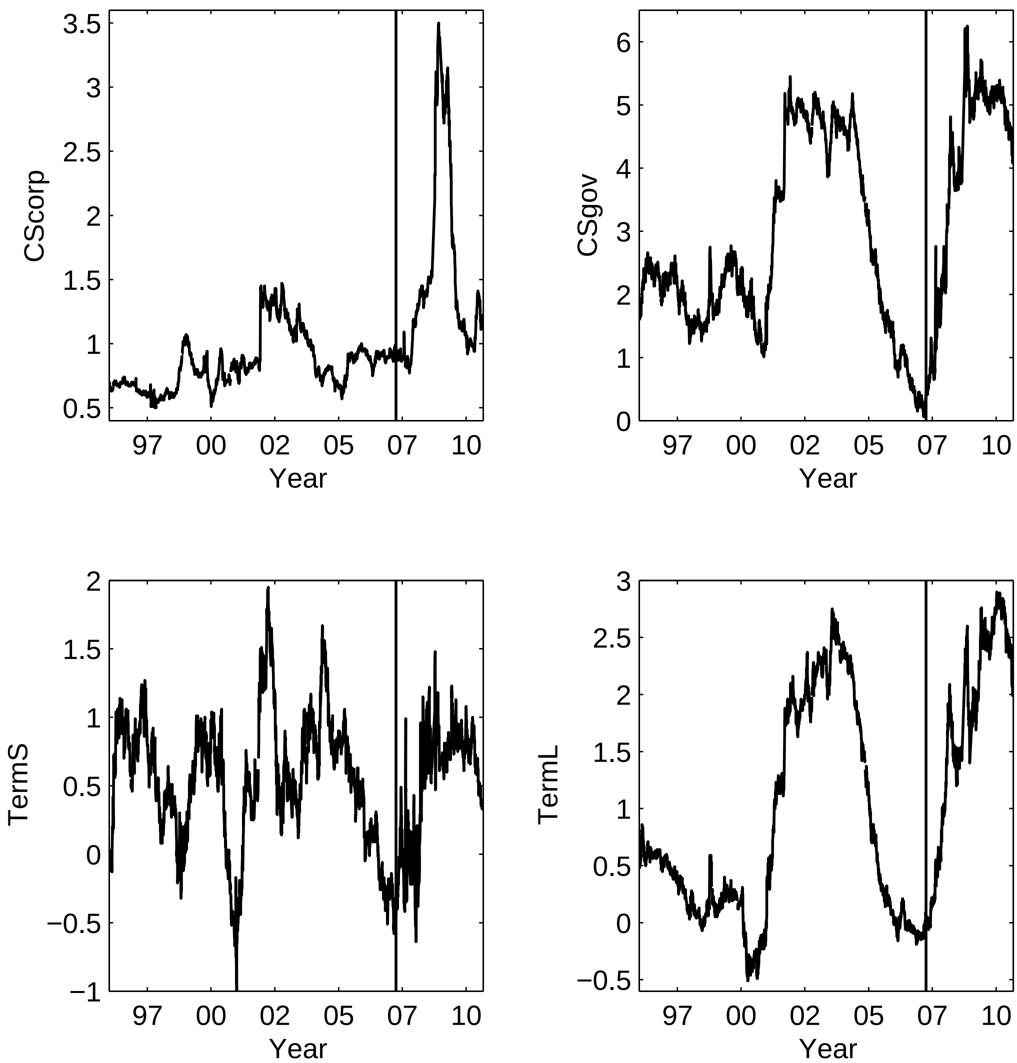
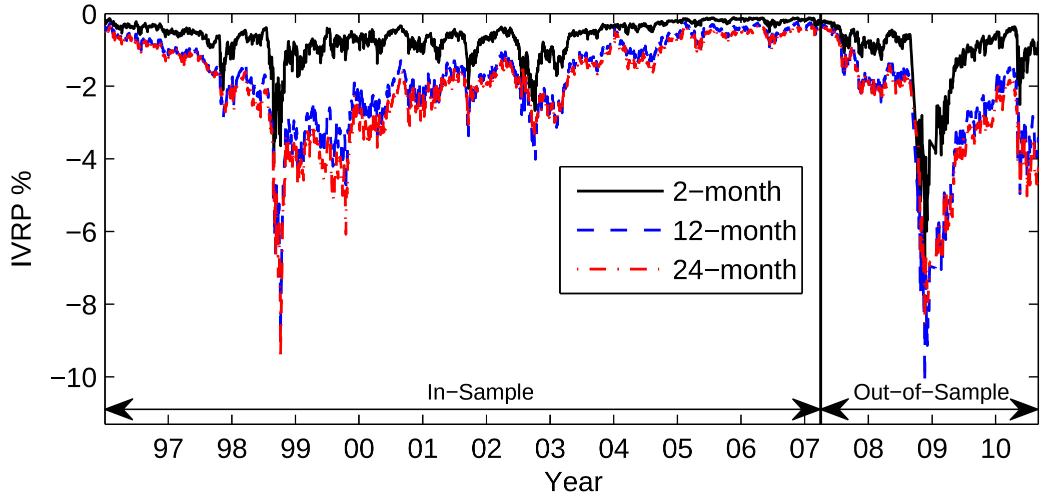


Figure 4. Time series plots of macro variables: CScorp the difference between Moody's BAA and AAA corporate bond yields, CSgov the difference between Moody's AAA corporate bond yield and 3-month Treasury securities, TermS the difference between the yields on 2-year and 3-month Treasury securities, TermL the difference between the yields on 10-year and 2-year Treasury securities. All variables are daily. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.



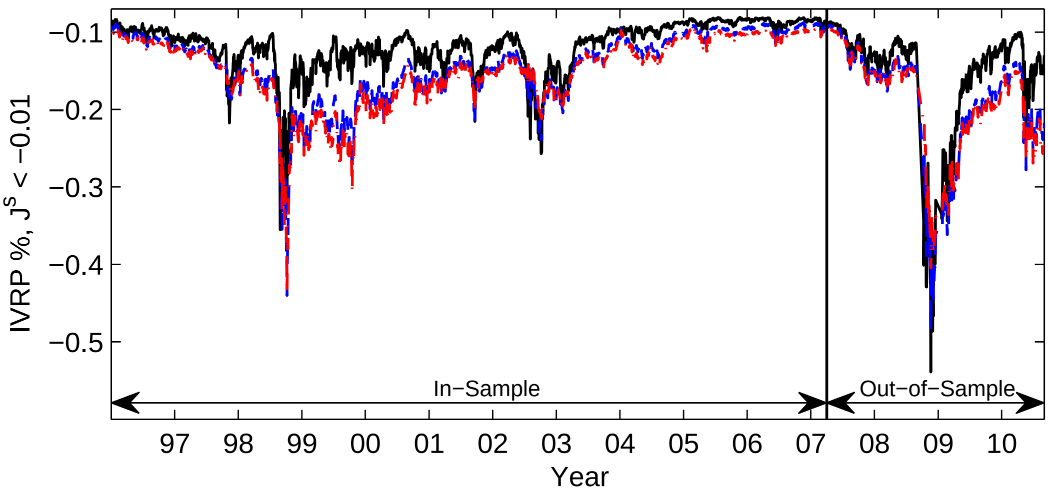


Figure 5. Term structure of integrated variance risk premia. Upper graph: integrated variance risk premia, i.e., $(E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}]) \times 100$. Lower graph: integrated variance risk premia due to price jump below $k = -0.01$, i.e., $(E_t^P[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}] - E_t^Q[\mathrm{QV}^j_{t,t+\tau}\, 1\{J^s < k\}]) \times 100$. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.

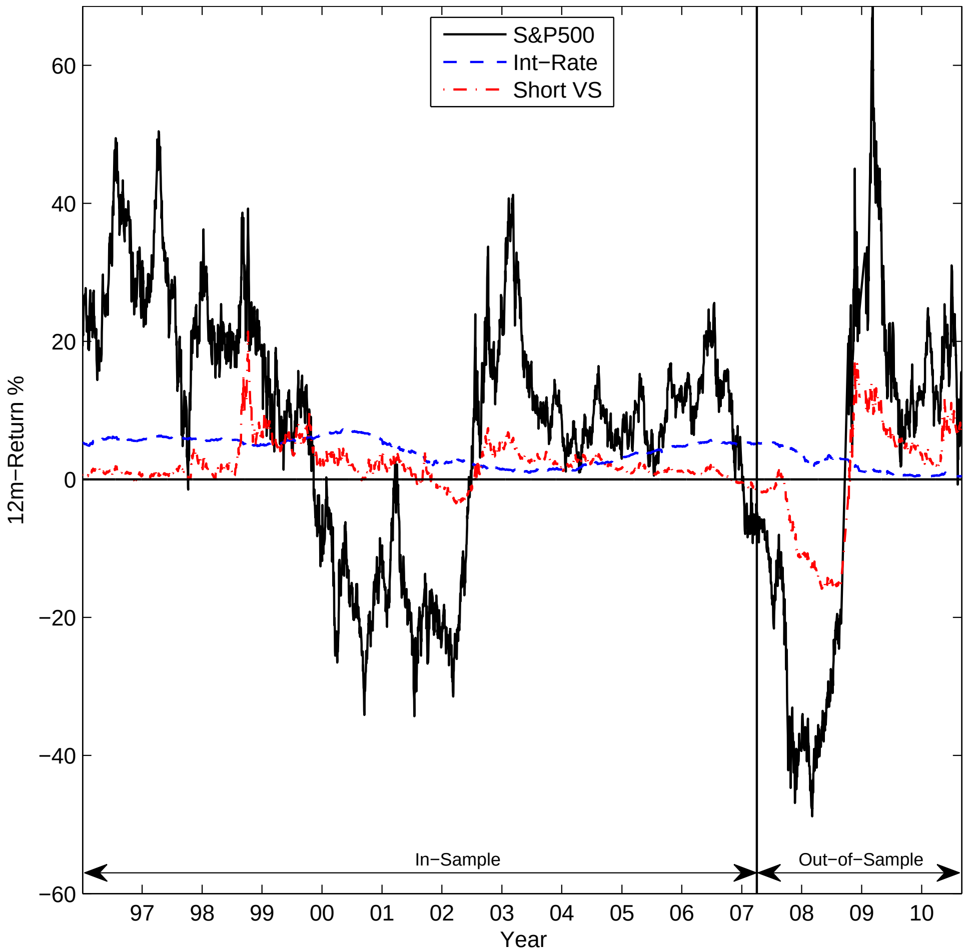


Figure 6. Returns of short positions in variance swap and long positions in the S&P500 index. Short VS (dash-dot line) denotes ex-post annual returns of the short-and-hold trading strategy based on 12-month VS, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$ for each day t in our sample, where τ is one year. S&P500 (solid line) denotes ex-post annual returns of the long-and-hold position on the S&P500 index, i.e., $S_{t+\tau}/S_t - 1$ for each day t in our sample, where τ is one year. Int-Rate (dash line) denotes the one-year interest rate for each day in our sample. Vertical line denotes beginning of out-of-sample period, i.e., April 3, 2007.

Maturity	Mean	Std	Skew	Kurt	AC1	Q_{22}	ADF
Panel A: Variance Swap Rates							
2	22.14	8.18	1.53	7.08	0.982	62,908.97	−3.79
3	22.32	7.81	1.32	6.05	0.988	66,449.22	−3.52
6	22.87	7.40	1.10	4.97	0.992	69,499.72	−3.30
12	23.44	6.88	0.80	3.77	0.994	71,644.69	−2.82
24	23.93	6.48	0.57	2.92	0.995	72,878.68	−2.47
Panel B: VIX-type Indices, CBOE method							
2	21.74	7.63	1.53	7.22	0.985	63,551.35	−3.83
3	21.95	7.32	1.43	6.83	0.987	64,644.41	−3.72
6	22.08	6.85	1.16	5.58	0.991	65,736.38	−3.62
Panel C: VIX-type Indices, Carr–Wu method							
2	22.34	7.82	1.53	7.19	0.985	63,699.78	−3.83
3	22.34	7.46	1.42	6.78	0.989	65,703.56	−3.83
6	22.30	7.00	1.20	5.75	0.999	66,490.72	−3.70
Panel D: Realized Variances							
2	18.09	8.62	2.13	10.70	0.997	68,750.50	−4.96
3	18.21	8.47	2.13	10.43	0.998	73,156.42	−4.73
6	18.58	8.37	2.04	9.07	0.999	76,928.13	−3.44
12	19.07	7.88	1.54	5.97	0.999	78,412.59	−2.51
24	19.91	6.97	0.68	3.02	0.999	76,028.01	−1.97

Table 1. Summary statistics, variables in levels. Panel A: variance swap rates on the S&P500 index. Time to maturities are in months. The sample period is from January 4, 1996 to September 2, 2010, for a total of 3,624 observations for each time to maturity. The table reports mean, standard deviation (Std), skewness (Skew), kurtosis (Kurt), first order autocorrelation (AC1) the Ljung–Box portmanteau test for up to 22nd order autocorrelation (Q_{22}), the test 10% critical value is 30.81; the augmented Dickey–Fuller test for unit root involving 22 augmentation lags, a constant term and time trend (ADF), the test 10% critical value is -3.16. Panels B and C: 2-, 3-, and 6-month VIX-type indices calculated using SPX options and applying the revised CBOE VIX and Carr–Wu methodologies, respectively. Panel D: ex-post S&P500 realized variances for various time to maturities. All variables are in volatility percentage units.

Maturity	Mean	Std	Skew	Kurt	AC1	Q_{22}	ADF
Panel A: Changes in Variance Swap Rates							
2	0.01	1.42	0.10	15.04	−0.11	166.09	−14.05
3	0.00	1.11	−0.01	14.87	−0.09	143.78	−13.97
6	0.00	0.83	0.42	12.61	−0.06	73.78	−13.42
12	0.00	0.65	0.29	12.05	−0.04	70.40	−13.18
24	0.00	0.53	0.18	10.94	−0.01	51.45	−13.41
Panel B: Changes in VIX-type Indices, CBOE method							
2	0.01	1.16	0.31	17.27	−0.12	114.07	−13.93
3	0.00	1.07	0.27	16.69	−0.09	101.00	−14.36
6	0.00	0.80	0.48	11.60	−0.01	73.14	−14.00
Panel C: Changes in VIX-type Indices, Carr–Wu method							
2	0.01	1.18	0.31	15.98	−0.11	126.48	−14.02
3	0.00	0.97	0.63	13.43	−0.06	76.78	−14.34
6	0.00	0.81	0.60	12.78	−0.03	83.17	−13.73

Table 2. Summary statistics, daily changes. Panel A: daily change in variance swap rates on the S&P500 index. Panels B and C: daily change in the 2-, 3-, and 6-month VIX-type indices calculated using SPX options and applying the revised CBOE VIX and Carr–Wu methodologies, respectively. Data, sample period and summary statistics are the same as in Table 1.

	SV1F		SV2F		SV2F-PJ		SV2F-PJ-VJ	
	Estim.	S.E.	Estim.	S.E.	Estim.	S.E.	Estim.	S.E.
κ_v^P	0.797	0.008	5.060	0.005	4.803	0.353	5.340	0.406
σ_v	0.272	0.002	0.525	0.003	0.419	0.009	0.394	0.006
κ_m^P			0.221	0.011	0.234	0.086	0.491	0.039
σ_m			0.154	0.002	0.141	0.002	0.167	0.001
θ_m^P	0.047	0.001	0.054	0.001	0.043	0.016	0.038	0.009
ρ	−0.674	0.008	−0.743	0.006	−0.713	0.010	−0.688	0.008
γ_1	1.303	2.537	0.742	2.591	−2.545	4.206	−5.054	5.495
γ_2	−1.322	1.173	−1.838	1.374	−2.244	0.851	−5.633	2.016
γ_3			−0.548	1.012	−0.673	0.610	−0.954	1.294
λ_0					3.669	0.621	2.096	0.467
λ_1					44.770	17.227	21.225	18.584
μ_j^P					0.010	0.008	−0.004	0.001
μ_j^Q					−0.001	0.009	−0.012	0.001
σ_j					0.038	0.003	0.043	0.000
μ_v^P							0.001	0.000
μ_v^Q							0.002	0.000
σ_{e_1}	0.006	0.000	0.004	0.000	0.004	0.000	0.004	0.000
σ_{e_2}	0.006	0.000	0.002	0.000	0.002	0.000	0.002	0.000
σ_{e_3}	0.011	0.000	0.003	0.000	0.003	0.000	0.007	0.000
σ_{e_4}	0.014	0.000						
ρ_e	0.288	0.016	−0.093	0.005	−0.088	0.006	−0.053	0.000
Log-likelihood	60,008.4		73,274.5		74,381.8		74,490.5	

Table 3. Model estimates. Estimation results for the Model (6)–(7) (labeled SV2F-PJ-VJ) and three nested models (labeled SV1F, SV2F and SV2F-PJ, respectively). For each model, estimate (Estim.) and standard errors (S.E.) are reported. The likelihood-based estimation procedure is described in Section 3. Variance swap rates with 2-, 3-, 6-, 12-, 24-month to maturity and S&P500 returns range from January 4, 1996 to April 2, 2007. Variance swap rates with 3- and 12-month (3-month) to maturity are assumed to be observed without errors (for the SV1F model). Variance swap rates with 2-, 6-, 24-month (and, for the SV1F model, 12-month) to maturity are assumed to be observed with errors whose standard deviations are σ_{e_1}, σ_{e_2}, σ_{e_3} (and σ_{e_4}), respectively, and correlation ρ_e. Interest rate $r = 4\%$ and dividend yield $\delta = 1.5\%$.

	In-Sample				Out-of-Sample			
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
	Heston		SJSV		Heston		SJSV	
$\widehat{\mathrm{VS}}_{2\mathrm{m}} - \mathrm{VS}_{2\mathrm{m}}$	−0.081	0.851	−0.151	0.739	0.259	1.420	0.194	1.036
$\widehat{\mathrm{VS}}_{6\mathrm{m}} - \mathrm{VS}_{6\mathrm{m}}$	0.002	1.119	0.058	0.397	−0.258	1.403	−0.209	0.470
$\widehat{\mathrm{VS}}_{24\mathrm{m}} - \mathrm{VS}_{24\mathrm{m}}$	1.001	2.950	0.144	0.555	0.469	3.074	−0.137	0.559

Table 4. Variance swap pricing errors. The pricing error is defined as the model-based VS rate minus observed VS rate, in volatility percentage units, i.e., $(E_t^Q[\mathrm{QV}_{t,t+\tau}]^{1/2} - \mathrm{VS}_{t,t+\tau}^{1/2}) \times 100$. The table reports mean and root mean square error of pricing errors for VS rate with 2-, 6-, and 24-month to maturity, under the Heston model and Model (6)–(7). In-sample period, used to estimate the models, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

	In-Sample		Out-of-Sample	
	Mean	Std	Mean	Std
DRP	4.66	3.99	9.30	9.49
JRP	2.38	0.57	3.05	1.36
VRP	−8.56	7.33	−17.08	17.42
LRMRP	−0.77	0.56	−1.20	0.68

Table 5. Spot risk premia. Diffusive risk premium $\text{DRP}_t = (\gamma_1(1-\rho^2) + \gamma_2\rho)v_t$; Jump risk premium $\text{JRP}_t = (E^P[e^J] - E^Q[e^J])(\lambda_0 + \lambda_1 v_t)$; Variance risk premium $\text{VRP}_t = \gamma_2\sigma_v v_t$; Long run mean risk premium $\text{LRMRP}_t = \gamma_3\sigma_m m_t$. Risk premia are based on Model (6)–(7). In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010. Entries are in percentage.

	In-Sample		Out-of-Sample		In-Sample		Out-of-Sample	
Maturity	Mean	Std	Mean	Std	Mean	Std	Mean	Std
	Equity				Variance			
2	6.68	3.71	11.00	8.29	−0.63	0.47	−1.18	1.06
6	6.37	3.08	9.63	5.81	−1.23	0.87	−2.21	1.72
12	6.28	2.79	8.82	4.43	−1.59	1.11	−2.71	1.83
24	6.45	2.52	8.28	3.40	−1.79	1.22	−2.89	1.74
	Price Jump Contribution				$J^s < -1\%$ Contribution			
2	2.40	0.49	2.97	1.11	−0.12	0.03	−0.16	0.07
6	2.44	0.44	2.89	0.84	−0.13	0.04	−0.18	0.08
12	2.52	0.43	2.88	0.68	−0.14	0.05	−0.19	0.07
24	2.71	0.46	2.97	0.59	−0.15	0.05	−0.19	0.07

Table 6. Term structure of integrated equity risk premia and integrated variance risk premia. Left panels: integrated equity risk premium, i.e., $(E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau) \times 100$, and equity risk premium due to the price jump component. Right panels: integrated variance risk premium, i.e., $(E_t^P[\text{QV}_{t,t+\tau}] - E_t^Q[\text{QV}_{t,t+\tau}]) \times 100$, and variance risk premium due to price jump J^s below -1%. Risk premia are based on Model (6)–(7). In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

Mat.	Interc.	S&P	VIX	CScorp	CSgov	TermS	TermL	R^2
			Panel A: Integrated Equity Risk Premium					
2	−5.27	−20.22	0.59	1.99	−1.20	1.04	0.96	93.1
	(−10.33)	(−3.25)	(14.79)	(3.11)	(−3.45)	(2.72)	(2.25)	
3	−4.56	−16.06	0.53	1.71	−0.67	0.70	0.29	92.5
	(−10.33)	(−2.79)	(13.75)	(2.91)	(−1.84)	(1.79)	(0.63)	
6	−3.10	−8.21	0.41	1.15	0.33	0.07	−0.95	88.2
	(−7.35)	(−1.63)	(10.28)	(2.06)	(0.62)	(0.13)	(−1.35)	
12	−1.57	−1.96	0.31	0.65	1.07	−0.39	−1.91	79.2
	(−2.88)	(−0.44)	(6.94)	(1.05)	(1.44)	(−0.56)	(−1.86)	
24	0.01	1.20	0.23	0.26	1.39	−0.54	−2.37	70.9
	(0.02)	(0.31)	(5.31)	(0.42)	(1.73)	(−0.73)	(−2.12)	
			Panel B: Integrated Variance Risk Premium					
2	0.88	2.72	−0.08	−0.26	0.18	−0.14	−0.16	93.3
	(13.36)	(3.50)	(−15.44)	(−3.21)	(4.20)	(−3.07)	(−3.11)	
3	1.13	3.13	−0.10	−0.31	0.17	−0.15	−0.13	93.3
	(14.48)	(3.16)	(−14.86)	(−3.10)	(2.92)	(−2.35)	(−1.76)	
6	1.52	2.89	−0.12	−0.36	−0.02	−0.05	0.14	90.6
	(14.12)	(2.06)	(−11.80)	(−2.44)	(−0.12)	(−0.42)	(0.87)	
12	1.65	1.06	−0.13	−0.29	−0.41	0.17	0.67	81.4
	(7.84)	(0.61)	(−7.39)	(−1.24)	(−1.41)	(0.64)	(1.68)	
24	1.46	−0.88	−0.11	−0.19	−0.73	0.37	1.09	70.3
	(4.71)	(−0.48)	(−4.98)	(−0.60	(−1.71)	(0.95)	(1.83)	

Table 7. Regression analysis for integrated risk premiums. Panel A: regression analysis of the annualized integrated equity risk premium, i.e., $(E_t^P[S_{t+\tau}/S_t]/\tau - E_t^Q[S_{t+\tau}/S_t]/\tau) \times 100$, based on Model (6)–(7). For each maturity (Mat.), the integrated equity risk premium is regressed on a constant (Interc.), S&P500 returns, VIX index, CScorp the difference between Moody's BAA and AAA corporate bond yields, CSgov the difference between Moody's AAA corporate bond yield and 3-month Treasury securities, TermS the difference between the yields on 2-year and 3-month Treasury securities, TermL the difference between the yields on 10-year and 2-year Treasury securities. All variables are daily. Maturity is in months. The sample period ranges from January 4, 1996 to September 2, 2010. For each maturity, the first row reports point estimates, the second row reports (in parenthesis) t-statistics based on robust standard errors computed using the Newey and West (1987) covariance matrix estimator with the number of lags optimally chosen according to Andrews (1991). R^2 is the adjusted R^2 in percentage. Panel B: corresponding regression analysis for the integrated variance risk premia, i.e., $(E_t^P[\mathrm{QV}_{t,t+\tau}] - E_t^Q[\mathrm{QV}_{t,t+\tau}]) \times 100$.

	In-Sample									
	Short Variance Swap					Long S&P500				
Horizon	2	3	6	12	24	2	3	6	12	24
Threshold										
Always	0.59	0.61	0.68	0.85	0.67	0.13	0.16	0.22	0.27	0.18
0	0.59	0.61	0.68	0.85	0.68	0.13	0.16	0.22	0.27	0.18
1/4	0.67	0.72	0.71	0.93	1.22	0.07	0.14	0.15	0.22	0.10
1/2	0.94	1.39	1.30	1.02	1.37	0.59	0.85	0.55	−0.03	−0.18
1	1.47	3.16	2.05	2.21	2.64	1.04	2.92	1.32	0.61	−0.33
	Out-of-Sample									
	Short Variance Swap					Long S&P500				
Always	0.23	0.17	0.08	0.03	0.07	−0.02	−0.10	−0.10	−0.18	−0.06
0	0.23	0.17	0.08	0.03	0.08	−0.02	−0.10	−0.10	−0.18	−0.05
1/4	0.67	0.21	0.10	0.04	0.26	0.36	−0.03	−0.10	−0.14	0.12
1/2	0.57	1.13	0.36	0.06	0.43	0.34	0.46	0.19	−0.09	0.29
1	0.32	0.76	1.84	2.47	2.98	0.09	0.11	1.20	1.33	1.51

Table 8. Sharpe ratios of short positions in variance swaps and long positions in the S&P500 index. For each day t in the sample, the expected profit from a short position in a VS contract is computed, i.e., $\mathrm{VS}_{t,t+\tau} - E_t^P[\mathrm{QV}_{t,t+\tau}]$. If the expected profit is n times larger than its standard deviation, then the VS contract is shorted. Otherwise no position is taken at day t. The column "Threshold" reports the number of standard deviations n. "Always" means the VS contract is always shorted. At time $t+\tau$, the actual profit is computed, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$, where $\mathrm{RV}_{t,t+\tau}$ is the ex-post realized variance. The notional amount in the VS contract is such that for each unit increase of the variance payoff, the contract pays out \$1. The investment strategy in the S&P500 is as follows. If at day t the VS contract with maturity $t+\tau$ is shorted, \$1 is invested in the S&P500 at day t. The position is held until $t+\tau$ and then liquidated. Sharpe ratios are computed using all the returns from each investment strategy. Interest rates are obtained by linearly interpolating the daily term structure of zero-coupon Treasury bond yields. VS contracts with 2-, 3-, 6-, 12- and 24-month to maturities are considered. The row "Horizon" reports the time to maturity. In-sample period, used to estimate the model, ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

	In-Sample			Out-of-Sample		
	S&P500	Int-Rate	Short VS	S&P500	Int-Rate	Short VS
	2-month Returns			2-month Returns		
S&P500	1.00	0.09	0.57	1.00	-0.31	0.63
Int-Rate	0.09	1.00	-0.01	-0.31	1.00	-0.25
Short VS	0.57	-0.01	1.00	0.63	-0.25	1.00
	12-month Returns			12-month Returns		
S&P500	1.00	0.05	0.30	1.00	-0.64	0.91
Int-Rate	0.05	1.00	-0.01	-0.64	1.00	-0.54
Short VS	0.30	-0.01	1.00	0.91	-0.54	1.00

Table 9. Correlations between returns of short positions in variance swaps, long positions in the S&P500 index and interest rates. Short VS denotes actual, ex-post returns of the short-and-hold VS position, i.e., $\mathrm{VS}_{t,t+\tau} - \mathrm{RV}_{t,t+\tau}$ for each day t in our sample, where τ is 2- and 12-month. S&P500 denotes actual, ex-post returns of the long-and-hold S&P500 position, i.e., $S_{t+\tau}/S_t - 1$ for each day t in our sample, where τ is 2- and 12-month. Int-Rate denotes the annualized interest rate for 2- and 12-month time horizons observed at a daily frequency. In-sample period ranges from January 4, 1996 to April 2, 2007. Out-of-sample period ranges from April 3, 2007 to September 2, 2010.

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