

Corporate Debt Value, Bond Covenants, and Optimal Capital Structure

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ABSTRACT

This article examines corporate debt values and capital structure in a unified analytical framework. It derives *closed-form* results for the value of long-term risky debt and yield spreads, and for optimal capital structure, when firm asset value follows a diffusion process with constant volatility. Debt values and optimal leverage are explicitly linked to firm risk, taxes, bankruptcy costs, risk-free interest rates, payout rates, and bond covenants. The results elucidate the different behavior of junk bonds versus investment-grade bonds, and aspects of asset substitution, debt repurchase, and debt renegotiation.

THE VALUE OF CORPORATE debt and capital structure are interlinked variables. Debt values (and therefore yield spreads) cannot be determined without knowing the firm's capital structure, which affects the potential for default and bankruptcy. But capital structure cannot be optimized without knowing the effect of leverage on debt value.

This article examines corporate debt values and optimal capital structure in a unified analytical framework. It derives *closed-form* results relating the value of long-term corporate debt and optimal capital structure to firm risk, taxes, bankruptcy costs, bond covenants, and other parameters when firm asset value follows a diffusion process with constant volatility.

Traditional capital structure theory, pioneered by Modigliani and Miller (1958), holds that taxes are an important determinant of optimal capital structure.¹ As leverage increases, the tax advantage of debt eventually will be offset by an increased cost of debt, reflecting the greater likelihood of financial distress.² While identifying some prime determinants of optimal capital

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¹Personal as well as corporate taxes will affect the tax benefits to leverage (Miller (1977)). Disagreement remains as to the precise value of net tax benefits.

²The costs of financial distress include bankruptcy costs and agency problems associated with risky debt. See, for example, Altman (1984), Asquith, Gertner, and Sharfstein (1991), Harris and Raviv (1991), Jensen and Meckling (1976), Myers and Majluf (1984), Titman and Wessels (1988), and Warner (1977).

structure, this theory has been less useful in practice because it provides qualitative guidance only.³

Brennan and Schwartz (1978) provide the first quantitative examination of optimal leverage. They utilize numerical techniques to determine optimal leverage when a firm's unlevered value follows a diffusion process with constant volatility.⁴ Although an important beginning, the Brennan and Schwartz analysis has three limitations.

First and most importantly, their numerical approach precludes general closed-form solutions for the value of risky debt and optimal leverage. Numerical examples suggest some possible comparative static results but cannot claim generality.

Second, their analysis focuses on the special case in which bankruptcy is triggered when the firm's asset value falls to the debt's principal value. This provision approximates debt with a positive net-worth covenant. But it is by no means the only—or even the typical—situation.⁵ We shall show that alternative bankruptcy-triggering conditions, including endogenously determined ones, lead to very different debt values and optimal capital structure.

Finally, Brennan and Schwartz (1978) consider changes in financial structure that last only until the bonds mature. A maturity date is required for their numerical algorithm; permanent capital structure changes are not explicitly analyzed.⁶

This article considers two possible bankruptcy determinants. The first is when bankruptcy is triggered (endogenously) by the inability of the firm to raise sufficient equity capital to meet its current debt obligations. The second is the Brennan and Schwartz case with a positive net-worth covenant. Debt with such a covenant will be termed *protected debt*.

We can derive closed-form results by examining corporate securities that depend on underlying firm value but are otherwise time independent. Yet debt securities generally have a specified maturity date and therefore have time-dependent cash flows and values. Time independence nonetheless can be justified, perhaps as an approximation, in at least two ways. First, if debt has sufficiently long maturity, the return of principal effectively has no value and

³Baxter (1967), Kraus and Litzenberger (1973), and Scott (1976) offer general analyses balancing tax advantages with the costs of financial distress, but their results have not provided directly usable formulas to determine optimal capital structure. For an alternative view on the determinants of capital structure, see Myers (1984).

⁴Kim (1978) also presents numerical examples of optimal capital structure, based on a mean-variance model. His model is less parsimonious, as knowledge of the joint distribution of market and firm returns is required.

⁵Minimum net-worth requirements are not uncommon in short-term debt contracts, but are rare in long-term debt instruments (also see Smith and Warner (1979)). In a later and more complex model, Brennan and Schwartz (1984) offer some examples with alternative bankruptcy conditions.

⁶Brennan and Schwartz do look at some examples when T becomes large. The relative insensitivity of these examples to T , as T exceeds 25 years, suggests that our limiting closed-form results for infinite maturity debt will be good approximations for debt with long but finite maturity.

can be ignored.⁷ Very long time horizons for fixed obligations are not new, either in theory or in practice. The original Modigliani and Miller (1958) argument assumes debt with infinite maturity. Merton (1974) and Black and Cox (1976) look at infinite maturity debt in an explicitly dynamic model. Since 1752 the Bank of England has, on occasion, issued Consols, bonds promising a fixed coupon with no final maturity date. And preferred stock typically pays a fixed dividend without time limit.

An alternative time-independent environment is when, at each moment, the debt matures but is rolled over at a fixed interest rate (or fixed premium to a reference risk-free rate) unless terminated because of failure to meet a minimum value, such as a positive net-worth covenant. As we discuss later, this environment bears resemblance to some revolving credit agreements.

Time independence permits the derivation of closed-form solutions for risky debt value, given capital structure. These results extend those of Merton (1974) and Black and Cox (1976) to include taxes, bankruptcy costs, and protective covenants (if any). They are then used to derive closed-form solutions for optimal capital structure.⁸ The analysis addresses the following questions:

- How do yield spreads on corporate debt depend on leverage, firm risk, taxes, payouts, protective covenants, and bankruptcy costs?
- Do high-risk (“junk”) bond values behave in qualitatively different ways than investment-grade bond values?
- What is the optimal amount of leverage, and how does this depend on risk-free interest rates, firm risk, taxes, protective covenants, and bankruptcy costs?

⁷For 30-year debt, the final repayment of principal represents 1.5 percent of debt value when the interest rate is 15 percent, and 5.7 percent of value when the interest rate is 10 percent. Recently, a number of firms have issued 50-year debt, and one firm (Disney) has issued 100-year debt.

⁸Recently I have become aware of important related work by Anderson and Sundaresan (1992), Longstaff and Schwartz (1992), and Mella and Perraudin (1993). Anderson and Sundaresan (1992) focus on risky debt in a binomial framework. Using numerical examples, they examine the choice of debtors to discontinue coupon payments prior to bankruptcy and show that this may explain the sizable default premiums found in bond prices (see Jones, Mason, and Rosenfeld (1984), and Sarig and Warga (1989)). They do not examine optimal capital structure.

Longstaff and Schwartz (1992) derive solutions for risky debt values with finite maturity and with stochastic risk-free interest rates. Their key assumption is that bankruptcy is triggered whenever firm value, V , falls to an exogenously given level, K , (our V_B), which is time independent. This is a strong assumption for finite maturity debt, whose debt service payments are time dependent. Equation (14) below shows that V_B depends on the risk-free interest rate, suggesting that an endogenously determined K should depend upon the (stochastic) interest rate. Longstaff and Schwartz (1992) do not consider optimal capital structure.

Mella and Perraudin’s approach more closely parallels this article, with endogenously determined bankruptcy levels. However, firm value is driven by a random product selling price whose drift as well as volatility must be specified, as must the firm’s cost structure. (See also Fries, Miller, and Perraudin (1993)). Like Anderson and Sundaresan (1992), the article considers an endogenous decision to continue service debt.

- How does a positive net-worth covenant affect the potential for agency problems between bondholders and stockholders?
- When can debt renegotiation be expected prior to bankruptcy, and can renegotiation achieve results that debt repurchase cannot?

The model follows Modigliani and Miller (1958), Merton (1974), and Brennan and Schwartz (1978) in assuming (i) that the activities of the firm are unchanged by financial structure, and (ii) that capital structure decisions, once made, are not subsequently changed.

Much of the recent literature in corporate finance examines possible variants to assumption (i): see, for example, the survey by Harris and Raviv (1991). A particularly important variant is the “asset substitution” problem, where shareholders of highly leveraged firms may transfer value to themselves from bondholders by choosing riskier activities. If the appropriate functional form were known, feedback from capital structure to volatility could be captured in an extension of our model, at the likely cost of losing closed-form results.⁹ But a simpler model that ignores such potential feedback still serves some important purposes:

- 1) Taxes and bankruptcy costs will importantly condition optimal capital structure even if asset substitution can occur; knowing these relationships in a basic model will provide useful insights for more complex situations.
- 2) The potential magnitude of the asset substitution problem can be identified by knowing how sensitive debt and equity values are to the risk of the activities chosen.
- 3) Bond covenants may directly limit opportunities for firms to alter the risk of their activities. In other cases, bond covenants may indirectly limit asset substitution by reducing potential conflicts of interest between stockholders and bondholders. Section VII below shows that a positive net worth requirement can eliminate the firm’s incentive to increase risk.

Our second major assumption is that the face value of debt, once issued, remains static through time. This is not as unreasonable as it might appear. In Section VIII, we show that additional debt issuance will hurt current debtholders; it is typically proscribed by bond covenants. We further show that marginal debt reductions via repurchases will hurt current stockholders. These considerations may preclude continuous changes in the outstanding amount of debt, even if refinancing costs are zero.

⁹Mello and Parsons (1992), using a numerical approach similar to Brennan and Schwartz (1978) but including operating decisions of a (mining) firm, contrast decisions that maximize equity value with those that maximize the total value of the firm. They associate the difference in resulting values with agency costs and present an example showing the effect of these costs on optimal leverage. Mauer and Triantis (1993) also use the Brennan and Schwartz (1978) approach to examine the interaction of investment decisions and corporate financing policies.

However, large (discontinuous) debt repurchases via tender offers *may* under certain circumstances benefit both stock and bondholders, if refinancing costs are not excessive. A dynamic model of capital structure capturing these possibilities is desirable but considerably more difficult. First steps in this direction have been made in important work by Kane, Marcus, and McDonald (1984) and Fischer, Heinkel, and Zechner (1989). Their analyses pose several difficulties, which we avoid by adopting the static assumption shared with earlier authors.¹⁰

The structure of the article is as follows. Section I develops a simple dynamic model of a levered firm, and derives values for time-independent securities. Sections II and III consider debt value and optimal leverage when bankruptcy is determined endogenously. Sections IV and V consider debt value and optimal leverage when bankruptcy is triggered by a positive net-worth covenant. Section VI considers some alternative assumptions about tax deductibility, cash payouts by the firm, and the absolute priority of payments in bankruptcy. Section VII addresses agency problems and asset substitution, while Section VIII considers aspects of debt repurchase and renegotiation. Section IX concludes.

I. A Model of Time-Independent Security Values

Consider a firm whose activities have value V which follows a diffusion process with constant volatility of rate of return:

$$dV/V = \mu(V, t)dt + \sigma dW, \quad (1)$$

where W is a standard Brownian motion. We shall refer to V as the “asset value” of the firm.¹¹ The stochastic process of V is assumed to be unaffected by the financial structure of the firm. Thus any net cash outflows associated with the choice of leverage (e.g., coupons after tax benefits) must be financed by selling additional equity.¹²

Following Modigliani and Miller (1957), Merton (1974), Black and Cox (1976), and Brennan and Schwartz (1978), we assume that a riskless asset

¹⁰ In Fischer, Heinkel, and Zechner (1989), the value of an unlevered firm (their A) cannot be exogenous, since it depends on the optimally levered firm value less costs of readjustment (see their p. 25). Since closed-form solutions are not available for the restructuring boundaries, they do not offer closed-form equations for risky debt value and optimal capital structure.

¹¹ We leave unanswered the delicate question of whether V , which could be associated with the value of an unlevered firm, is a traded asset. An alternative approach is to note that if equity, E , is a traded security, its process could be used to *define* a process, V , through equation (13) below, using Ito’s Lemma. Our assumption that V has constant volatility will restrict the permissible process of E .

¹² This is consistent with bond covenants that restrict firms from selling assets. Brennan and Schwartz (1978) also make this assumption, although Merton (1974) does not. In Section VI.B, we consider how our results are affected by relaxing this assumption.

exists that pays a constant rate of interest r . This permits us to focus on the *risk structure* of interest rates directly.¹³

Now consider any claim on the firm that continuously pays a nonnegative coupon, C , per instant of time when the firm is solvent. Denote the value of such a claim by $F(V, t)$. When the firm finances the net cost of the coupon by issuing additional equity, it is well known (e.g., Black and Cox (1976)) that any such asset's value must satisfy the partial differential equation

$$(1/2)\sigma^2V^2F_{VV}(V, t) + rVF_V(V, t) - rF(V, t) + F_t(V, t) + C = 0 \quad (2)$$

with boundary conditions determined by payments at maturity, and by payments in bankruptcy should this happen prior to maturity.¹⁴ In general, there exist no closed-form solutions to equation (2) for arbitrary boundary conditions. Hence Brennan and Schwartz (1978) resort to computer analysis of some examples. However, when securities have no explicit time dependence, the term $F_t(V, t) = 0$ and equation (2) becomes an ordinary differential equation with $F(V)$ satisfying

$$(1/2)\sigma^2V^2F_{VV}(V) + rVF_V(V) - rF(V) + C = 0. \quad (3)$$

Equation (3) has the general solution

$$F(V) = A_0 + A_1V + A_2V^{-X}, \quad (4)$$

where

$$X = 2r/\sigma^2 \quad (5)$$

and the constants A_0 , A_1 , and A_2 are determined by boundary conditions. Any time-independent claim with an equity-financed constant payout $C \geq 0$ must have this functional form. We turn now to examining specific securities.

Debt promises a perpetual coupon payment, C , whose level remains constant unless the firm declares bankruptcy. The value of debt can be expressed as $D(V; C)$. For simplicity, however, we will suppress the coupon as an argument and simply write debt value as $D(V)$. Let V_B denote the level of asset value at which bankruptcy is declared. (Note that we again suppress the argument C .) If bankruptcy occurs, a fraction $0 \leq \alpha \leq 1$ of value will be

¹³Extensions of numerical bond valuation to include interest rate risk are provided in Brennan and Schwartz (1980) and Kim, Ramaswamy, and Sundaresan (1993). They find that the yield spreads between corporate and Treasury bonds are quite insensitive to interest rate uncertainty.

¹⁴More generally, if net payouts by the firm not financed by further equity issuance are denoted $P(V, t)$, and $C(V, t)$ represents the payout flow to security F , then

$$(1/2)\sigma^2(V, t)V^2F_{VV}(V, t) + [rV - P(V, t)]F_V(V, t) - rF(V, t) + F_t(V, t) + C(V, t) = 0.$$

Note that $\sigma^2(V, t)$ could be of the form $\sigma^2[C(V, t), V, t]$, reflecting possible asset substitution.

Equation (2) requires that V , or an asset perfectly correlated (locally) with V , such as equity, be traded. See also footnote 11.

lost to bankruptcy costs, leaving debtholders with value $(1 - \alpha)V_B$ and stockholders with nothing.¹⁵

Later we show how the bankruptcy value, V_B , is determined, given alternative debt covenants. For the moment regard it as fixed. Since the value of debt is of the form in equation (4), we must determine the constants A_0 , A_1 , and A_2 . Boundary conditions are:

$$\text{At } V = V_B, \quad D(V) = (1 - \alpha)V_B \tag{6i}$$

$$\text{As } V \rightarrow \infty, \quad D(V) \rightarrow C/r. \tag{6ii}$$

Condition (6ii) holds because bankruptcy becomes irrelevant as V becomes large, and the value of debt approaches the value of the capitalized coupon (and therefore the value of risk-free debt).

From equation (4), it is immediately apparent using equation (6ii) that $A_1 = 0$. Because $V^{-X} \rightarrow 0$ as $V \rightarrow \infty$, this with equation (6ii) implies that $A_0 = C/r$. Finally, $A_2 = [(1 - \alpha)V_B - C/r]V_B^X$, using equation (6i). Thus

$$D(V) = C/r + [(1 - \alpha)V_B - C/r][V/V_B]^{-X}. \tag{7}$$

Equation (7) can also be written as $D(V) = [1 - p_B](C/r) + p_B[(1 - \alpha)V_B]$, where $p_B \equiv (V/V_B)^{-X}$ has the interpretation of the present value of \$1 contingent on future bankruptcy (i.e., V falling to V_B).¹⁶

Equation (7) represents a straightforward extension of Black and Cox (1976) to include bankruptcy costs.¹⁷ But we shall see later that taxes affect the value, V_B , when bankruptcy is determined endogenously. Both taxes and bankruptcy costs are important determinants of debt value in this case.

Debt issuance affects the total value of the firm in two ways. First, it reduces firm value because of possible bankruptcy costs. Second, it increases firm value due to the tax deductibility of the interest payments, C . The value of both these effects will depend upon the level of firm value, V , and are time independent. Therefore they can be valued as if they were time-independent securities.

First, consider a security that pays no coupon, but has value equal to the bankruptcy costs αV_B at $V = V_B$. This security has current value, denoted

¹⁵We focus on bankruptcy costs that are proportional to asset value when bankruptcy is declared. Alternatives such as constant bankruptcy costs could readily be explored within the framework developed. Deviations from absolute priority (in which bondholders do not receive all remaining value) can also be incorporated in the boundary conditions; we do so in Section VI.C. Franks and Torous (1989) and Eberhart, Moore, and Roenfeldt (1990) document deviations from the absolute priority rule.

¹⁶More exactly,

$$p_B = \int_0^\infty \exp(-rt)f(t; V, V_B) dt,$$

where $f(t; V, V_B)$ is the density of the first passage time from V to V_B , when the process for V has drift equal to the risk-free interest rate, r .

¹⁷Merton (1974) derives a different formula for the case where $\alpha = 0$. This is because he assumes the firm liquidates assets to pay coupons.

$BC(V)$, that reflects the market value of a claim to αV_B should bankruptcy occur. Because its returns are time independent, it too must satisfy equation (4) with boundary conditions

$$\text{At } V = V_B, \quad BC(V) = \alpha V_B \quad (8i)$$

$$\text{As } V \rightarrow \infty, \quad BC(V) \rightarrow 0 \quad (8ii)$$

In this case equation (4) has solution

$$BC(V) = \alpha V_B (V/V_B)^{-X}. \quad (9)$$

BC is a decreasing, strictly convex function of V . Again, note the reinterpretation of equation (9) as $BC = p_B[\alpha V_B]$: the current value of bankruptcy costs is their magnitude if bankruptcy occurs, times the present value of \$1 conditional on future bankruptcy. Subsequent expressions will have similar interpretations.

Now consider the value of tax benefits associated with debt financing. These benefits resemble a security that pays a constant coupon equal to the tax-sheltering value of interest payments (τC) as long as the firm is solvent and pays nothing in bankruptcy. This security's value, $TB(V)$, equals the value of the tax benefit of debt. It too is time independent and therefore must satisfy equation (4) with boundary conditions

$$\text{At } V = V_B, \quad TB(V) = 0 \quad (10i)$$

$$\text{As } V \rightarrow \infty, \quad TB(V) = \tau C/r. \quad (10ii)$$

Equation (10i) reflects the loss of the tax benefits if the firm declares bankruptcy. Equation (10ii) reflects the fact that, as bankruptcy becomes increasingly unlikely in the relevant future, the value of tax benefits approaches the capitalized value of the tax benefit flow, τC . Using equation (4) and the boundary conditions above gives

$$TB(V) = \tau C/r - (\tau C/r)(V/V_B)^{-X}. \quad (11)$$

Tax benefits are an increasing, strictly concave function of V .

Note that the value of tax benefits, equation (11), presumes that the firm *always* benefits fully (in amount τC) from the tax deductibility of coupon payments when it is solvent. But under U.S. tax codes, to benefit fully the firm must have earnings before interest and taxes (EBIT) that is at least as large as the coupon payment, C .¹⁸ An alternative approach, in which EBIT is related to asset value, V , and tax benefits may be lost when the firm is solvent (but close to bankruptcy), is considered in Section VI.A.

¹⁸The losses associated with interest payments exceeding EBIT may be carried forward, but lose time value, and may lose all value if the firm goes bankrupt. (Reorganizations under Chapter 11 of the Bankruptcy Code may carry forward some tax benefits. This could be modeled by a boundary condition, equation (10i), with a positive value.)

The total value of the firm, $v(V)$, reflects three terms: the firm's asset value, plus the value of the tax deduction of coupon payments, less the value of bankruptcy costs:

$$v(V) = V + TB(V) - BC(V) \\ = V + (\tau C/r) \left[1 - (V/V_B)^{-X} \right] - \alpha V_B (V/V_B)^{-X}. \quad (12)$$

Note that v is strictly concave in asset value, V , when $C > 0$ and either $\alpha > 0$ or $\tau > 0$. Note also that if $\alpha > 0$ and $\tau > 0$, then $v(V) < V$ as $V \rightarrow V_B$, and $v(V) > V$ as $V \rightarrow \infty$. This coupled with concavity implies that v is (proportionately) more volatile than V at low values of V and is less volatile at high values.

The value of equity is the total value of the firm less the value of debt:

$$E(V) = v(V) - D(V) \\ = V - (1 - \tau)C/r + [(1 - \tau)C/r - V_B][V/V_B]^{-X}. \quad (13)$$

We see from equation (14) below that when V_B is endogenously determined, $[(1 - \tau)C/r - V_B] > 0$, implying that $E(V)$ is a convex function of V . This reflects the "option-like" nature of equity, even when debt has an infinite horizon. When V_B is determined by a positive net worth requirement, however, we show in Section V that equity may be a *concave* function of V . This has important ramifications for agency problems associated with asset substitution, which are examined in Section VII. Finally, Ito's Lemma can be used to show that the volatility of equity's rate of return declines as V (and therefore E) rises. Stock option pricing models would need to reflect this nonconstant volatility, as well as the possibility that E reaches zero with positive probability.

Equations (7) and (13) indicate the importance of V_B in determining the values of debt and equity. In the following sections, we consider alternative bankruptcy-triggering scenarios.

II. Debt with No Protective Covenants: The Endogenous Bankruptcy Case

If the firm is not otherwise constrained by covenants, bankruptcy will occur only when the firm cannot meet the required (instantaneous) coupon payment by issuing additional equity: that is, when equity value falls to zero.¹⁹ However, *any* level of asset value, V_B , that triggers bankruptcy will imply

¹⁹In continuous time, the coupon (Cdt) paid over the infinitesimal interval, dt , is itself infinitesimal. Therefore the value of equity simply needs to be positive to avoid bankruptcy over the next instant. In discrete time, where the time between periods, δt , is of a fixed size, the value of equity at each period must exceed the coupon ($C\delta t$) to be paid that period.

It is sometimes assumed that bankruptcy is triggered by a cashflow shortage. This can be criticized, because, if equity value remains, a firm will always be motivated and able to issue additional equity to cover the shortage, rather than declare bankruptcy. Positive equity value rather than positive cashflow seems to be the essential element when bankruptcy is endogenously determined.

that the value of equity is zero at that asset value, given the absolute priority rule.

When V_B can be chosen by the firm (rather than imposed by a covenant such as positive net-worth requirement), it can be seen from equation (12) that total firm value, v , will be maximized by setting V_B as low as possible. Limited liability of equity, however, prevents V_B from being arbitrarily small: $E(V)$ must be nonnegative for all values of $V \geq V_B$. From equation (13), $E(V)$ is strictly convex in V when $V_B < (1 - \tau)C/r$. Thus the lowest possible value for V_B consistent with positive equity value for all $V > V_B$ is such that $dE/dV|_{V=V_B} = 0$: a "smooth-pasting" or "low contact" condition at $V = V_B$. (This choice of bankruptcy level can also be shown to maximize the value of equity at any level of V : $dE/dV_B = 0$ ²⁰). Differentiating equation (13) with respect to V , setting this expression equal to zero with $V = V_B$, and solving for V_B gives

$$V_B = [(1 - \tau)C/r][X/(1 + X)] = (1 - \tau)C/(r + 0.5\sigma^2), \quad (14)$$

where the second line uses equation (5). Since $V_B < (1 - \tau)C/r$, equity is indeed convex in V .

Observe that the asset value, V_B , at which bankruptcy occurs

- a) is proportional to the coupon, C ;
- b) is independent of the current asset value, V ;
- c) decreases as the corporate tax rate, τ , increases;
- d) is independent of bankruptcy costs, α ;
- e) decreases as the risk-free interest rate, r , rises; and
- f) decreases with increases in the riskiness of the firm, σ^2 .

The results above also describe the behavior of total firm value at bankruptcy, $v_B \equiv v[V_B] = (1 - \alpha)V_B$, except that v_B falls as bankruptcy cost, α , increases. The fact that asset value, V , does not affect v_B means that the bankruptcy level of total firm market value can be estimated from the coupon, C , (plus parameters r , σ_2 , α , and τ), without needing to know the firm's current asset value.²¹

Substituting equation (14) into equations (7), (12), and (13) gives

$$D(V) = (C/r)[1 - (C/V)^X k] \quad (15)$$

$$v(V) = V + (\tau C/r)[1 - (C/V)^X h] \quad (16)$$

$$E(V) = V - (1 - \tau)(C/r)[1 - (C/V)^X m] \quad (17)$$

²⁰See also Merton (1973; footnote 60). The equivalence of the two conditions suggests that the endogenously set V_B is incentive compatible in the following sense. Ex ante (before debt issuance), stockholders will wish to maximize firm value subject to the limited liability of equity. The ex ante optimal V_B achieves this by satisfying the smooth-pasting condition. Ex post, equity holders will have no incentive to declare bankruptcy at a different V , since V_B also satisfies the ex post optimal condition for maximizing equity value.

²¹Knowledge of the market value of equity, E , and debt, D , in addition to C , combined with equations (7), (13), and (14), permits calculation of a unique V and α given r , τ , and σ . Alternatively, α and σ can be recovered, given r , τ , and V .

where

$$\begin{aligned} m &= [(1 - \tau)X/r(1 + X)]^X/(1 + X) \\ h &= [1 + X + \alpha(1 - \tau)X/\tau]m \\ k &= [1 + X - (1 - \alpha)(1 - \tau)X]m. \end{aligned}$$

The interest rate paid by risky debt, $R(C/V)$, can be derived directly from dividing C by $D(V)$, giving

$$R(C/V) = C/D(V) = rK(C/V), \tag{18}$$

where

$$K(C/V) = [1 - (C/V)^X k]^{-1}.$$

The interest rate depends positively on the *ratio* of the coupon, C , to firm asset value, V . Note $K(C/V)$ has the interpretation of a risk-adjustment factor (multiplying the risk-free rate) that the firm must pay to compensate bondholders for the risks assumed. The yield spread is $R(C/V) - r = r(C/V)^X k/[1 - (C/V)^X k]$.

The values above are derived for an arbitrary level of the coupon, C . Section III examines the *optimal* choice of coupon (and leverage) for unprotected debt. But first, we examine the behavior of unprotected debt values and yield premiums for an arbitrary coupon level.

A. The Comparative Statics of Debt Value ($D(V)$)

Equation (15) extends Black and Cox's (1976) results to include the effects of taxes and bankruptcy on debt value. Row 1 of Table I summarizes the comparative statics of debt value. Not surprisingly, larger bankruptcy costs decrease the value of debt. Less obvious is that an increase in the corporate tax rate will raise debt value, through lowering the bankruptcy level, V_B .²²

More surprising still are the results when taxes or bankruptcy costs are positive and firm asset value, V , nears the bankruptcy level, V_B . Table I indicates that the effects of increases in the coupon, firm riskiness, and the risk-free rate become reversed from what is expected. An increase in coupon can *lower* debt value. An increase in firm risk can *raise* debt value, as can an increase in the risk-free rate. Thus *the behavior of "junk" bonds (or "fallen angels") differs significantly from the behavior of investment-grade bonds when bankruptcy costs and/or taxes are positive.*²³

To understand these results, first consider the presence of positive bankruptcy costs. If V is close to V_B , the value of debt will be very sensitive to such costs. Lowering V_B will raise the value of debt since bankruptcy costs

²²These comparative static results presume that other parameters (including V) remain at their current level, the usual *ceteris paribus* assumption. Note, however, that a change in the corporate tax rate might affect V as well.

²³The ratios of V/V_B (or C/V) at which the various behaviors are reversed are not identical. Of course, these ratios may not correspond to Wall Street's definition of "junk" bonds.

Table I
Comparative Statics of Financial Variables: Unprotected Debt

This table describes properties of the equations describing debt value, D , the interest, R , paid on debt, the yield spread of the debt over the risk-free rate ($R - r$), the total firm value, v , and the value of equity, E , when debt is *not* protected by a positive net-worth covenant. V is the firm's asset value, V_B is the endogeneously determined value at which bankruptcy is declared, C is the coupon paid on debt, σ^2 is the variance of the asset return, r is the risk-free interest rate, α is the fraction of asset value lost if bankruptcy occurs, and τ is the corporate tax rate.

Variable	Homogeneity	Shape	Limit As		Sign of Change in Instrument for an Increase in:								
			$V \rightarrow \infty$	$V \rightarrow V_B$	C	σ^2	r	α	τ	V			
D	Degree 1 in V, C	Concave in V, C	C/r	$\frac{C(1-\alpha)(1-\tau)}{(r+0.5\sigma^2)}$	> 0 ;	< 0 ;	< 0 ;	< 0 ;	< 0 ;	< 0	> 0	> 0	> 0
R	Degree 0 in V, C	Convex in V/C	r	$\frac{(r+0.5\sigma^2)}{(1-\alpha)(1-\tau)}$	$< 0^*$ as $V \rightarrow V_B$	$> 0^*$ as $V \rightarrow V_B$	> 0 ;	> 0 ;	> 0 ;	> 0	< 0	< 0	< 0
$R - r$	Degree 0 in V, C	Convex in V/C	0	$\frac{[0.5\sigma^2 + r(\alpha + \tau - \alpha\tau)]}{[(1-\alpha)(1-\tau)]}$	> 0	> 0 ;	> 0 ;	$< 0^*$ as $V \rightarrow V_B$	< 0	> 0	< 0	< 0	< 0
v	Degree 1 in V, C	Concave in V, C	$V + \tau C/r$	$\frac{C(1-\alpha)(1-\tau)}{(r+0.5\sigma^2)}$	> 0 ;	< 0 ;	< 0 ;	< 0 ;	< 0 ;	< 0	> 0	> 0	> 0
E	Degree 1 in V, C	Convex in V, C	$V - (1-\tau)C/r$	0	$< 0^*$ as $V \rightarrow V_B$	$> 0^*$ as $V \rightarrow V_B$	> 0	> 0	> 0	0	> 0	> 0	> 0

*Sign reversal as $V \rightarrow V_B$ only if α and/or $\tau > 0$.

will be less imminent. From equation (14), higher asset volatility, higher risk-free interest rates, or lower coupon, C , will all serve to lower V_B . For values of V close to V_B , this positive effect on $D(V)$ will dominate. Even if there are no direct bankruptcy costs, the event of bankruptcy causes the value of the tax shield to be lost when $\tau > 0$, and the previous conclusions continue to hold.

The fact that $D(V)$ is eventually decreasing as the coupon rises implies that debt value reaches a maximum, $D_{max}(V)$, for a finite coupon, $C_{max}(V)$. We can naturally think of D_{max} as the *debt capacity* of the firm. Differentiating equation (15) with respect to C , setting the resulting equation equal to zero and solving for C gives

$$C_{max}(V) = V[(1 + X)k]^{-1/X} \tag{19}$$

Substituting this into equation (15) and simplifying gives

$$D_{max}(V) = V[Xk^{-1/X}(1 + X)^{-(1+1/X)}]/r. \tag{20}$$

The debt capacity of a firm is proportional to asset value, V , and falls with increases in firm risk, σ^2 , and bankruptcy costs, α . Debt capacity rises with increases in the corporate tax rate, τ , and the risk-free rate, r .

Figures 1 and 2 show the relationship between debt value and the coupon

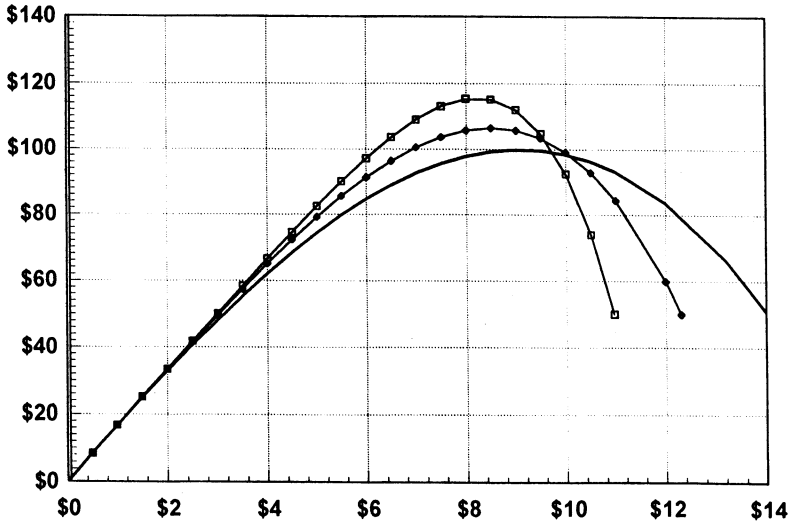


Figure 1. Debt value as a function of the coupon, when debt is unprotected. The lines plot the value of unprotected debt, D , at varying coupon levels C , for three levels of asset volatility, σ : 15 percent (open square), 20 percent (filled diamond), and 25 percent (solid line). It is assumed that the risk-free interest rate $r = 6.0\%$, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

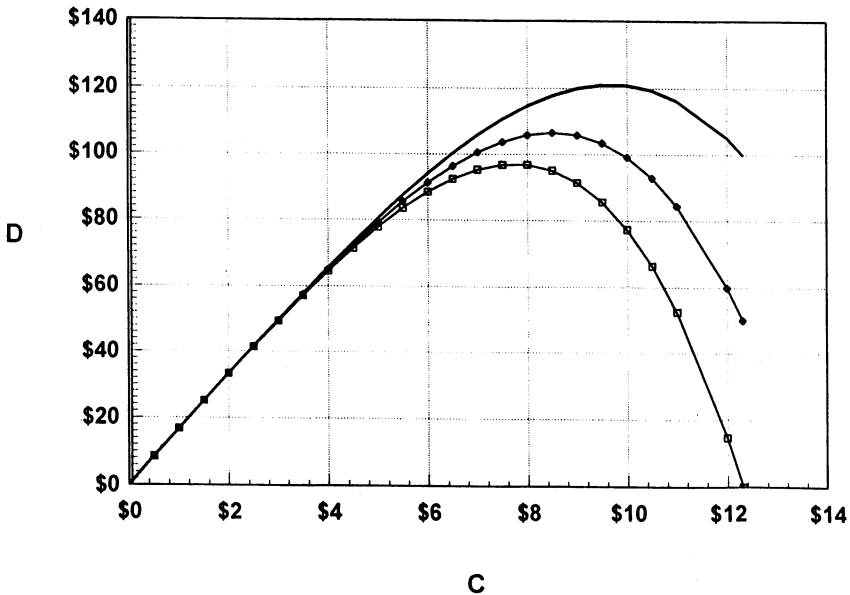


Figure 2. Debt value as a function of the coupon, when debt is unprotected. The lines plot the value of unprotected debt, D , at varying coupon levels, C , for three levels of bankruptcy costs: 0 (solid line), 50 percent (filled diamond), and 100 percent (open square) ($\alpha = 0, 0.5$, and 1, respectively). It is assumed that the risk-free interest rate $r = 6.0$ percent and the corporate tax rate is 35 percent ($\tau = 0.35$).

for varying firm volatility and bankruptcy costs, when $V = \$100$ and $r = 6$ percent. Our normalization implies that the coupon level (in dollars) also represents the coupon rate as a percentage of asset value, V . Note that at high coupon levels, the debt of riskier firms has higher value than that of less risky firms. The peak of each curve indicates the maximum debt capacity, D_{max} , with corresponding leverage level. Figure 3 repeats Figure 1, but with leverage, $[D/v]$, rather than coupon level on the x -axis. The reversals seen in Figure 1 do not appear in Figure 3. This is because leverage itself depends on the value of debt.

B. Yield Spreads: The Risk Structure of Interest Rates

Rows 2 and 3 of Table I indicate the behavior of risky interest rates and yield spreads. Increasing the coupon, C , always raises the yield spread. An increase in bankruptcy costs, α , also raises the spread, although a rise in the corporate tax rate will lower the spread because debt value will rise. Related to our earlier discussion, we observe the surprising result that *junk bond yield spreads may actually decline when firm riskiness increases*. Of course, this holds only for junk bonds: the yield spread on investment-grade debt will increase when firm risk rises. Also note that junk bond interest rates may

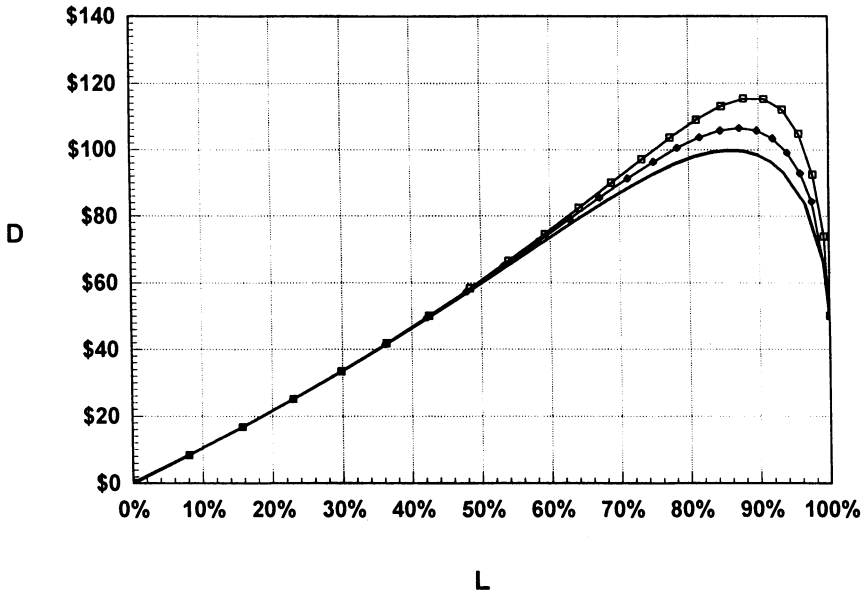


Figure 3. Debt value as a function of the leverage, when debt is unprotected. The lines plot the value of unprotected debt, D , at varying leverage ratios, L , for three levels of asset volatility, σ : 15 percent (*open square*), 20 percent (*filled diamond*), and 25 percent (*solid line*). It is assumed that the risk-free interest rate $r = 6.0\%$, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

actually fall when the risk-free rate increases. Figures 4 and 5 plot yield spreads against coupon level and leverage, respectively, as asset value risk changes.

Observe that $R(C/V) \rightarrow (r + 0.5\sigma^2)$ as $V \rightarrow V_B$ when $\alpha = \tau = 0$. That is, long-term risky debt will never have a yield exceeding the risk-free rate by more than $0.5\sigma^2$ if there are no bankruptcy costs or tax benefits to debt.²⁴ Observing a yield spread greater than this on corporate long-term debt implies the presence of bankruptcy costs, taxes, or both.²⁵

C. The Comparative Statics of Firm Value ($v(V)$) and Equity Value ($E(V)$)

Row 4 of Table I indicates the comparative statics of total firm value. Again observe the perverse behavior of total firm value for firms with junk debt. In the presence of bankruptcy costs and/or corporate taxes, total firm value

²⁴A firm whose asset value has an annual standard deviation of 20 percent, for example, would have debt whose yield spread *never* exceeds two percent. It has been argued that the tax advantage to debt may be nil (Miller (1977)). For arguments that bankruptcy costs may be small, see Warner (1977) (who focuses on direct costs only) and Haugen and Senbet (1988).

²⁵When the firm has several debt issues, junior debt could have higher rates. But the weighted average cost of debt will be limited to $r + 0.5\sigma^2$ in this case.

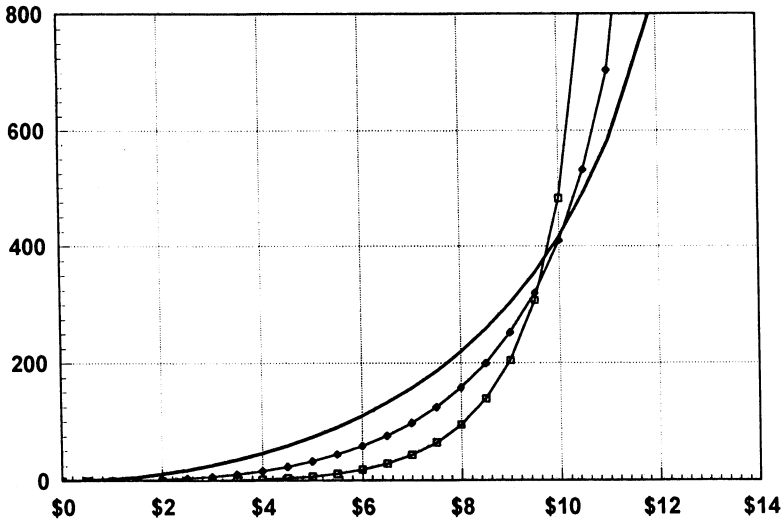


Figure 4. Yield spreads on unprotected debt as a function of the coupon. The lines plot the yield spread, YS (in basis points/year), the amount the firm's debt yield exceeds the risk-free rate, as a function of the coupon, C , for varying levels of asset volatility, σ : 15 percent (*open square*), 20 percent (*filled diamond*), and 25 percent (*solid line*). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

may rise as firm riskiness increases. Rising risk-free rates may also lead total firm value to increase. The values of firms with investment-grade debt will not exhibit such behavior. Figure 6 and Figure 7 illustrate total firm value, v , as a function of the coupon level C and the leverage D/v , respectively. Optimal leverage is the ratio at which each curve reaches its peak.

Row 5 of Table I indicates the behavior of equity value. Unlike debt, there are no reversals of comparative static results when V is close to V_B . The fact that bankruptcy costs do not affect equity value is perhaps surprising, but it reflects the fact that, given the coupon, C , debtholders bear all bankruptcy costs. In Section III we show that the *optimal* coupon and debt-equity ratio do depend upon α , and that initial equity holders ultimately are hurt by greater bankruptcy costs.

III. Optimal Leverage with Unprotected Debt

Consider now the coupon rate, C , which maximizes the total value, v , of the firm, given current asset value, V . Differentiating equation (16) with respect to C , setting the derivative equal to zero and solving for the optimal coupon,

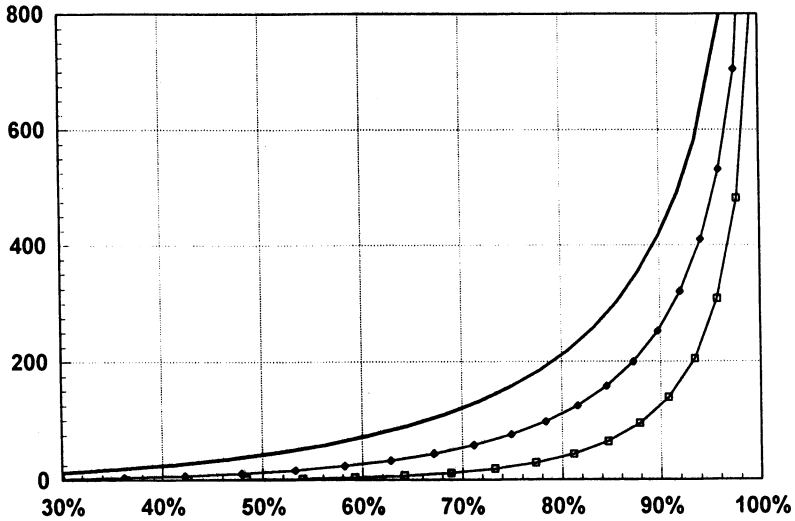


Figure 5. Yield spreads on unprotected debt as a function of the leverage. The lines plot the yield spread, *YS* (in basis points/year), the amount the firm's debt yield exceeds the risk-free rate, as a function of the leverage, *L*, for varying levels of asset volatility, σ : 15 percent (*open square*), 20 percent (*filled diamond*), and 25 percent (*solid line*). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

C^* , as a function of asset value, V , gives:

$$C^*(V) = V[(1 + X)h]^{-1/X} \tag{21}$$

Note that $h > k$, implying $C^*(V) < C_{max}(V)$. Substituting $C^*(V)$ into equations (15), (16), (18), and (14) gives

$$D^*(V) = V[(1 + X)h]^{-1/X} \{1 - k[(1 + X)h]^{-1}\} / r \tag{22}$$

$$v^*(V) = V\{1 + (\tau/r)[(1 + X)h]^{-1/X} [X/(1 + X)]\} \tag{23}$$

$$R^* = r[(1 + X)h] / [(1 + X)h - k] \tag{24}$$

$$V_B^*(V) = V(m/h)^{1/X} \tag{25}$$

Table II indicates the comparative statics of these variables plus optimal leverage $L^* = D^*/v^*$ and equity $E^* = v^* - D^*$. While most results are consistent with what is expected, a few merit comment.

The optimal coupon C^* is a U-shaped function of firm riskiness, as illustrated in Figure 8. Firms with little business risk, or very large risk, will optimally commit to pay sizable coupons. Firms with intermediate levels of risk will promise smaller coupons. However, the optimal leverage ratios of riskier firms will always be less than those of less risky firms, as can be seen

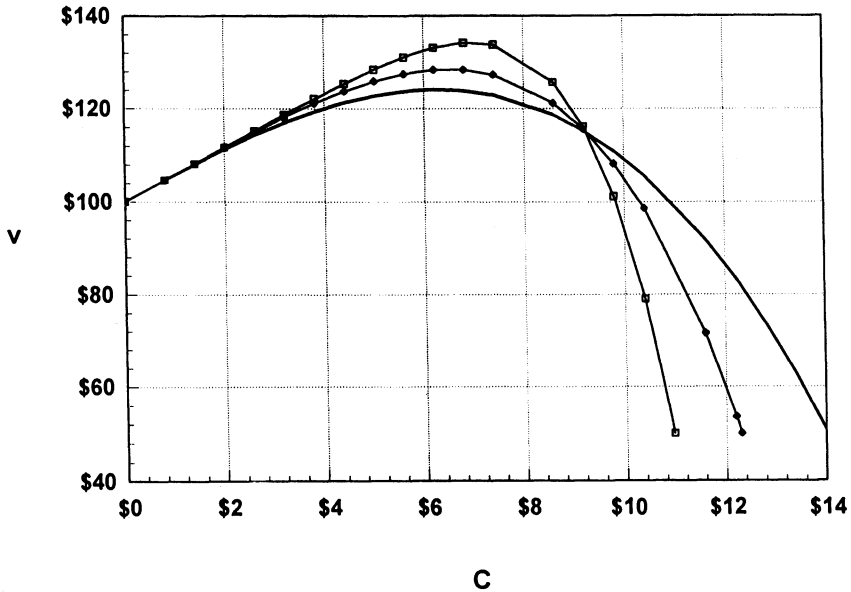


Figure 6. Total firm value as function of the coupon, when debt is unprotected. The lines plot total firm value, v , at varying coupon levels, C , for three levels of asset volatility, σ : 15 percent (*open square*), 20 percent (*filled diamond*), and 25 percent (*solid line*). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

by observing the maximal firm values in Figure 7. The potential gains in moving from no leverage to optimal leverage (where $v = v^*$) are considerable. For reasonable parameter levels, optimizing financial structure can increase firm value by as much as 25 to 40 percent over a firm with no leverage.

Our results confirm Brennan and Schwartz's (1978) observation that optimal leverage is less than 100 percent even when bankruptcy costs are zero. Too high leverage risks bankruptcy—and while there are no bankruptcy costs, the tax deductibility of coupon payments is lost.

Leverage of about 75 to 95 percent is optimal for firms with low-to-moderate levels of asset value risk and moderate bankruptcy costs.²⁶ Even firms with high risks and high bankruptcy costs should have leverage on the order of 50 to 60 percent, when the effective tax rate is 35 percent. Optimal

²⁶ It is of interest that many of the leveraged buyouts of the 1980s created capital structures that had 95 percent leverage or more. And targets were often firms with relatively stable value (low σ^2). Our analysis indicates these firms will reap maximal benefits from increased leverage. Subsequent leverage reduction by many of these firms could in part be explained by the substantial fall in interest rates, which reduces the optimal leverage ratio.

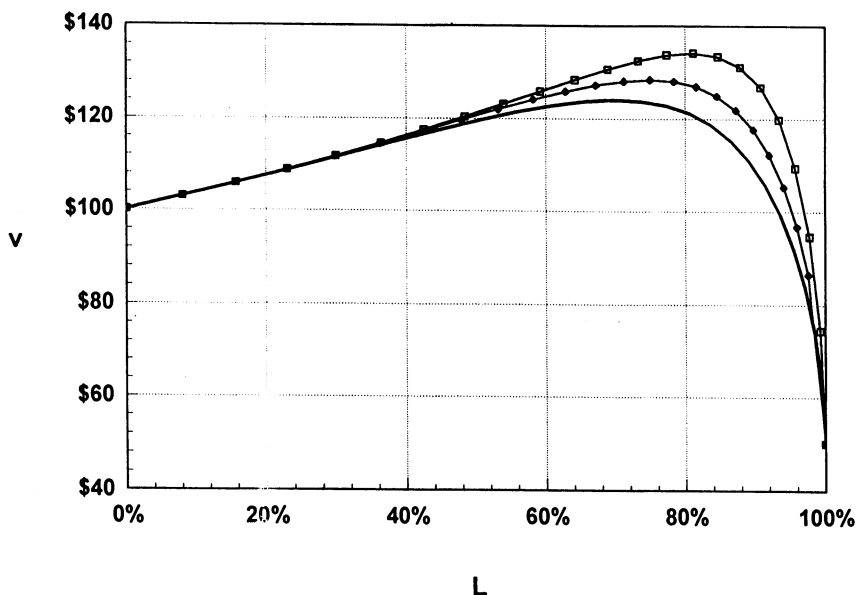


Figure 7. Total firm value as function of the leverage, when debt is unprotected. The lines plot total firm value, v , at varying levels of leverage L , for three levels of asset volatility, σ : 15 percent (*open square*), 20 percent (*filled diamond*), and 25 percent (*solid line*). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

leverage ratios drop by 5 to 25 percent when the effective tax rate is 15 percent, with the more pronounced falls at high volatility levels.²⁷ Variations of our assumptions that lead to lower optimal leverage ratios are discussed in Section VI.

The behavior of the yield spread at the optimal leverage ratio exhibits one surprise. Increased bankruptcy costs might be thought to increase interest rates. Indeed they do—but only if the coupon is fixed. As bankruptcy costs rise, the optimal coupon C^* falls. The probability of bankruptcy is then less and the yield spread decreases. Figure 9 illustrates yield spreads at the optimal leverage as a function of bankruptcy costs, α , and asset risk, σ .

Higher risk-free interest rates might also be expected to reduce the optimal amount of borrowing, but they do not: the added tax shield when interest rates are high more than offsets the greater costs of borrowing. This could be destabilizing, since supply would normally be expected to decrease as interest

²⁷ Following Miller (1977), if the effective personal tax rate on stock returns (reflecting tax deferral) were 20 percent, the tax rate on bond income were 40 percent, and the corporate tax rate 35 percent, the effective tax advantage of debt is $[1 - (1 - 0.35)(1 - 0.20)/(1 - 0.40)] = 0.133$, or slightly less than 15 percent.

Table II
Comparative Statics of Financial Variables at the Optimal Leverage Ratio: Unprotected Debt

This table describes the behavior of the coupon, C^* , that maximizes firm value, and the debt, D^* , leverage, L^* , interest rate, R^* , yield spread, $R^* - r$, total firm value, v^* , equity value, E^* , and bankruptcy value, V_B^* , at the optimal coupon level, for unprotected debt. (V is the firm's asset value, σ^2 is the variance of the asset return, r is the risk-free interest rate, α is the fraction of asset value lost if bankruptcy occurs, and τ is the corporate tax rate.

Variable	Shape	Sign of Change in Variable for an Increase in:			
		σ^2	r	α	τ
C^*	Linear in V	< 0 , σ^2 small; > 0 , σ^2 large	> 0	< 0	> 0
D^*	Linear in V	< 0	> 0	< 0	> 0
L^*	Invariant to V	< 0	> 0	< 0	> 0
R^*	Invariant to V	> 0	> 0	< 0	> 0
$R^* - r$	Invariant to V	> 0	< 0	< 0	> 0
v^*	Linear in V	< 0	> 0	< 0	> 0
E^*	Linear in V	> 0	< 0	> 0	$< 0^a$
V_B^*	Linear in V	< 0	> 0	< 0	$> 0^a$

^aNo effect if $\alpha = 0$.

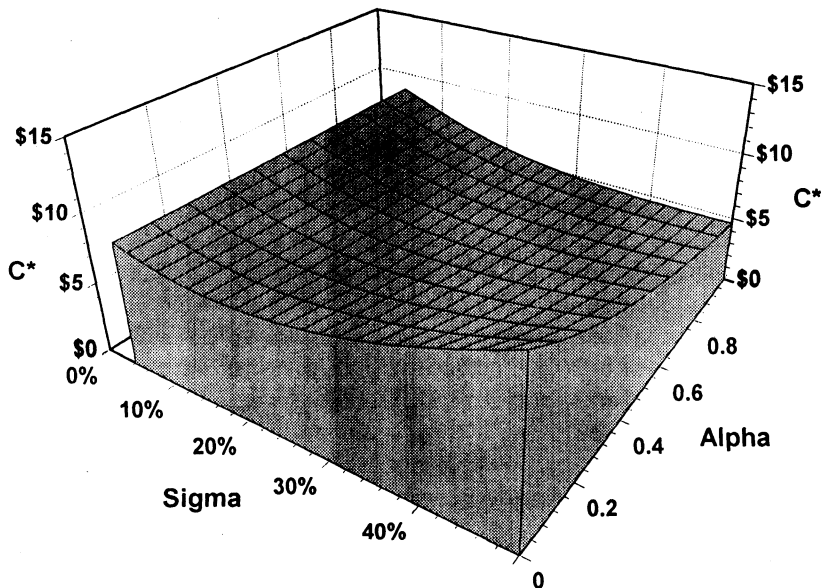


Figure 8. The optimal coupon as a function of firm risk and bankruptcy costs. The surface plots the optimal coupon, C^* , at varying levels of firm risk, σ , and bankruptcy costs, α . It is assumed that the risk-free interest rate $r = 6.0$ percent and the corporate tax rate is 35 percent ($\tau = 0.35$).

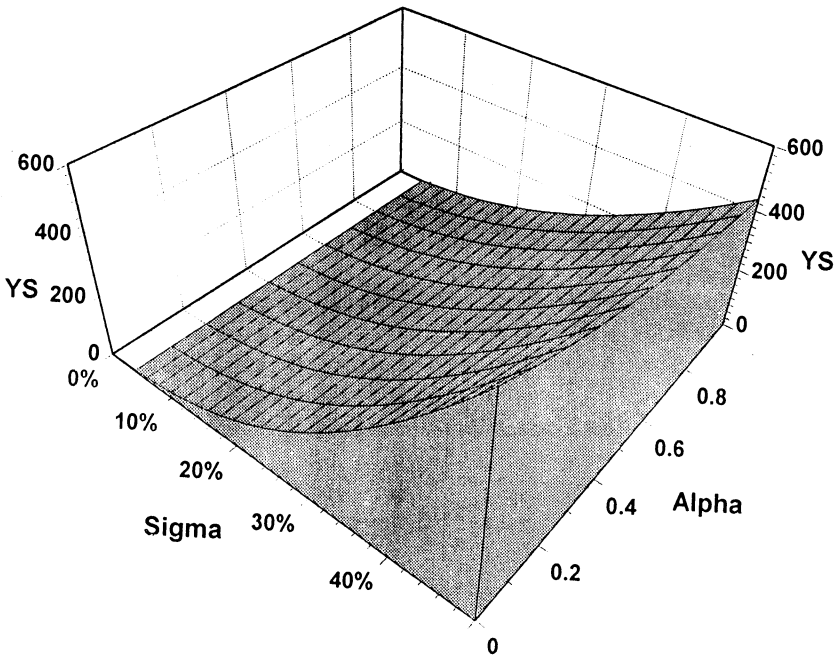


Figure 9. The yield spread as a function of firm risk and bankruptcy costs. The surface plots the yield spread, YS, the difference between the yield on the firm's debt (at the optimal coupon, C^*) and the risk-free interest rate, r , for varying levels of firm risk, σ , and bankruptcy costs, α . It is assumed that the risk-free interest rate $r = 6.0$ percent and the corporate tax rate is 35 percent ($\tau = 0.35$).

rates rise.²⁸ Despite the greater borrowing, the yield spread at the optimal leverage actually falls slightly as the risk-free interest rate increases.

IV. Positive Net-Worth Covenants and the Value of Protected Debt

Consider now the case in which debt remains outstanding without time limit unless bankruptcy is triggered by the value of the firm's assets falling beneath the principal value of debt, denoted P . We presume the principal value coincides with the market value of the debt when it is issued, denoted D_0 . Thus $V_B = D_0$.²⁹

²⁸Note that an increase in r might well cause a decline in V . If so, it is possible that the desired amount of borrowing (which is proportional to V) could decline even though optimal leverage rises.

²⁹It must be verified that the $V_B = D_0$ is consistent with the value of equity remaining positive at all levels $V \geq V_B$. This requires that D_0 exceed the level in equation (14) satisfying the smooth-pasting conditions. In fact, this is always the case at the optimal protected debt coupon level, and is satisfied at all but extremely high initial coupon levels. We limit our examination of protected debt to coupon levels for which the minimum net-worth requirement (rather than equation (14)) is the determinant of V_B .

Are there contractual arrangements in which this is a realistic description of bankruptcy? One possibility would be long-term debt as examined previously, with a protective covenant stipulating that the asset value of the firm always exceed the principal value of the debt: a positive net-worth requirement. Such covenants are not common in long-term bond contracts, however.

An alternative contractual arrangement approximating this case would be a continuously renewable line of credit, in which the borrowing amount and interest rate are fixed at inception.³⁰ At each instant the debt will be extended (“rolled over” at a fixed interest rate) if and only if the firm has sufficient asset value, V , to repay the loan’s principal, P ; otherwise bankruptcy occurs.³¹ Thus the roll-over process proxies for a positive net-worth requirement. With this latter interpretation, the differences between the unprotected debt analyzed above and protected debt analyzed below *may capture many of the differences between long-term debt and (rolled over) short-term financing.*

From equation (7) with $V_B = D_0$, we can write the value of protected debt as a function of the value of assets, V_0 , at the time the debt is initiated:

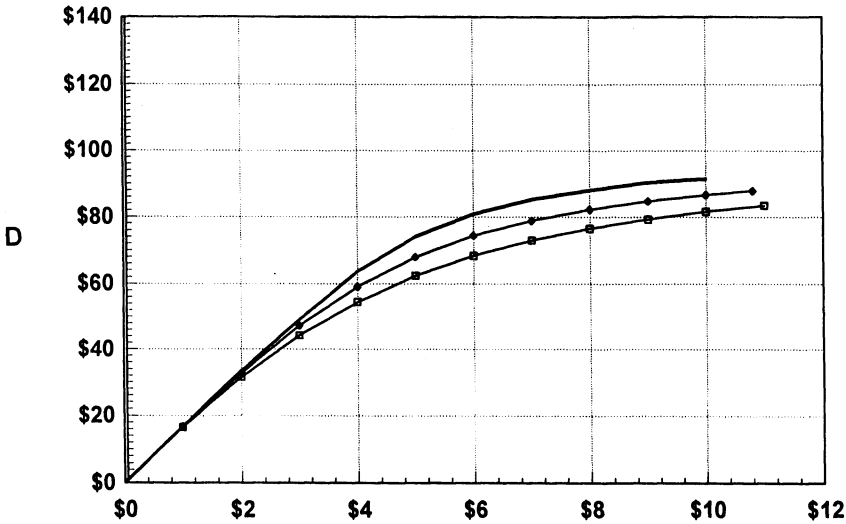
$$D_0(V_0) = C/r + [(1 - \alpha)D_0(V_0) - C/r][V_0/D_0(V_0)]^{-X} \quad (26)$$

Except when $\alpha = 0$, closed-form solutions for the function $D_0(V_0)$ satisfying equation (26) have not been found. However, we can easily solve this equation numerically to determine the value, D_0 , of the debt, given initial values, V_0 and C (as before we suppress the argument C). Note that the function $D_0(V_0)$ is homogeneous of degree one in V_0 and C . Also note that equation (26) gives the value of protected debt only at the initial asset value, V_0 . Equation (7) with $V_B = D_0(V_0)$ gives protected debt value as a function of asset value, V .

Figures 10 and 11 illustrate the behavior of protected debt value as the coupon and leverage change, for $V = V_0 = 100$. They should be compared with Figures 1 and 3. We observe that the surprising behavior of unprotected “junk” debt does not hold for protected debt, even when the debt exhibits considerable risk. Unlike the unprotected case, the value of debt increases with the coupon at all levels of C . And increased firm risk or a higher risk-free interest rate always lowers debt value. This is because the bankruptcy-triggering value, V_B , is determined exogenously rather than endogenously.

³⁰ We assume the firm will never choose to borrow less than the stipulated credit line amount. The fact that most credit lines are tied to a floating rate is not important here, since the risk-free rate is assumed to be constant. It is important that the interest rate paid by the firm be independent of the firm’s asset value V (providing $V \geq V_B$) after the initial agreement is reached.

³¹ Many lines of credit have a “paydown” provision, requiring that the amount borrowed must be reduced to zero at least once per year. A firm will fail to meet this provision if its (market) value of assets is less than the loan principal. Also note that Merton (1974) requires $V \geq P$ at maturity to avoid default on a pure-discount bond: the firm must have positive net worth at maturity or bankruptcy occurs.



C

Figure 10. Debt value as a function of the coupon, when debt is protected by a minimum net-worth requirement. The lines plot the value of protected debt, D , at varying coupon levels, C , for three levels of asset volatility, σ : 15 percent (solid line), 20 percent (filled diamond), and 25 percent (open square). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

When there are no bankruptcy costs ($\alpha = 0$),

- a) Protected debt is riskless and pays the risk-free rate, r .
- b) For any C , the value of the tax shield with protected debt is less than the tax shield with unprotected debt.
- c) For any C , the bankruptcy-triggering value of assets, V_B , for protected debt exceeds the V_B for unprotected debt.

Protected debt is riskless when $\alpha = 0$ because the firm's asset value is constantly monitored. Should asset value fall to the principal value, bankruptcy is declared and, because there are no bankruptcy costs, debtholders receive their full principal value. In this case, for a given coupon, C , the value of protected debt always exceeds that of unprotected debt. Further, $V_B = P = D_0(V_0) = C/r$. This exceeds the bankruptcy-triggering value, equation (14), of assets for unprotected debt, and implies smaller tax benefits from equation (11).

When bankruptcy costs are positive ($\alpha > 0$), the results change markedly. For a given coupon, C , protected debt may have a lesser value than unprotected debt (and therefore may pay a higher interest rate). This follows because bankruptcy will occur more frequently when debt is protected, because V_B is higher in the protected case, and bankruptcy costs will be

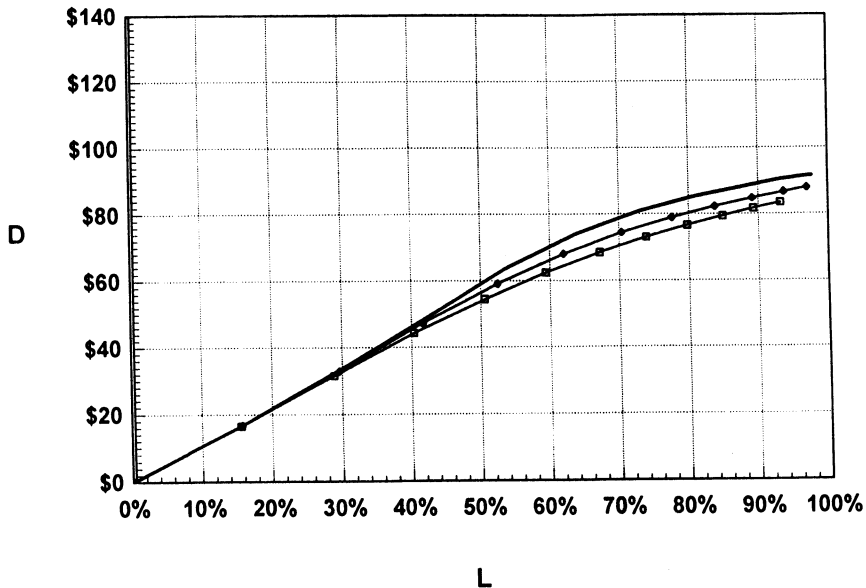


Figure 11. Debt value as a function of the leverage, when debt is protected by a minimum net-worth requirement. The lines plot the value of protected debt, D , at varying leverage ratios, L , for three levels of asset volatility, σ : 15 percent (*solid line*), 20 percent (*filled diamond*), and 25 percent (*open square*). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

incurred when $\alpha > 0$. Figure 12, when compared with Figure 5, shows yield spreads to be substantially higher for protected debt when $\alpha = 0.5$, except at extreme leverage ratios.

V. Optimal Leverage with Protected Debt

We now use a simple search procedure to find the coupon, C^* , that maximizes the total value, v , of the firm with protected debt. Figure 13, compared with Figure 7, illustrates that maximal firm value occurs at lower leverage when debt is protected.

For a reasonable range of parameters, we find that

- Optimal leverage for protected debt is substantially less than for unprotected debt.
- The interest rate paid at the optimum leverage is less for protected debt, even when bankruptcy costs are positive ($\alpha > 0$).
- The maximum value of the firm (and therefore the benefit from leverage) is less when protected debt is used.

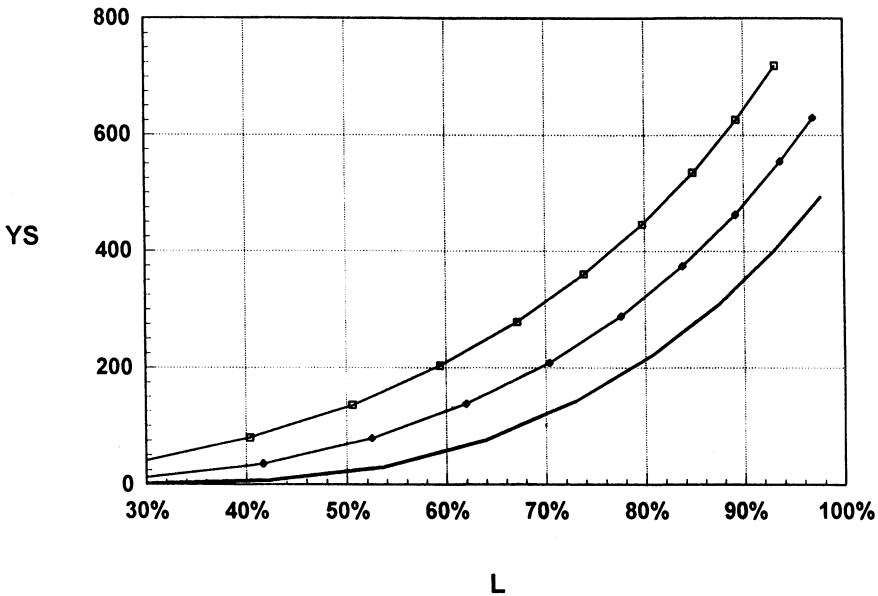


Figure 12. Yield spreads on protected debt as a function of the leverage. The lines plot the yield spread, *YS* (in basis points/year), the amount the firm's debt yield exceeds the risk-free rate, as a function of the leverage, *L*, for varying levels of asset volatility, σ : 15 percent (*solid line*), 20 percent (*filled diamond*), and 25 percent (*open square*). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35% ($\tau = 0.35$).

d) The maximal benefits of unprotected over protected debt increase as:

- Corporate taxes increase
- Interest rates are higher
- Bankruptcy costs are lower

A closer examination of numerical results reveals that *the optimal bankruptcy level V_B^* is the same for both protected and unprotected debt*, when bankruptcy costs are zero. We know, however, the closed-form solution for unprotected debt's optimal bankruptcy level, V_B , from equation (25). Since $D_0 = V_B$, this in turn suggests a closed-form solution for the optimal value of protected debt and related values when bankruptcy costs are zero and $V = V_0$:

$$D_0^*(V_0) = V_B^*(V_0) = V_0(m/h)^{1/X} \tag{27}$$

Because protected debt is risk free when $\alpha = 0$, it also follows that

$$C^*(V_0) = rD_0^*(V_0) = rV_B^*(V_0) = rV_0(m/h)^{1/X} \tag{28}$$

$$v^*(V_0) = V_0 + [\tau C^*(V_0)/r] \{1 - [C^*(V_0)/V_0]^X h\} \tag{29}$$

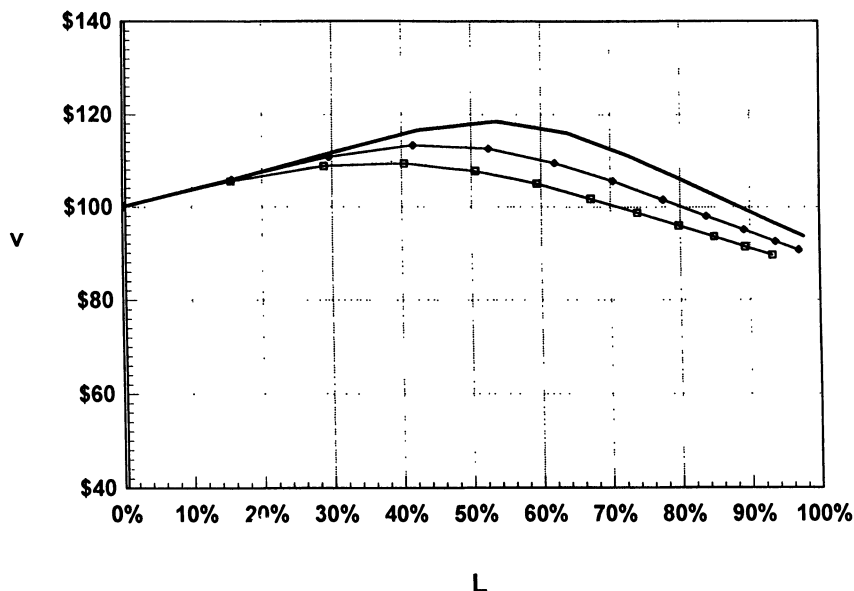


Figure 13. Total firm value as function of the leverage, when debt is protected. The lines plot total firm value, v , at varying levels of leverage, L , for three levels of asset volatility, σ : 15 percent (solid line), 20 percent (filled diamond), and 25 percent (open square). It is assumed that the risk-free interest rate $r = 6.0$ percent, bankruptcy costs are 50 percent ($\alpha = 0.5$), and the corporate tax rate is 35 percent ($\tau = 0.35$).

Recall that equations (27) to (29) hold only for the protected debt case with no bankruptcy costs. We have not been able to find closed-form solutions when $\alpha > 0$. Equation (28) implies that $[(1 - \tau)(C^*/r) - V_B^*] = -\tau C^*/r < 0$, when $\alpha = 0$. From equation (13), this implies that *equity is a strictly concave function of V* . By continuity, equity will be concave when α is close to zero. And in the numerical example considered in Section VI, equity is strictly concave in V for all α .

The observed comparative statics of optimal protected debt value (and related values) are given in Table III. There are some important differences with the comparative statics of optimal unprotected debt value. The debt yield and the yield spread at the optimum rise rather than fall as bankruptcy costs rise. The yield spread also increases as the risk-free interest rate rises, although the magnitude is small. The optimal leverage ratio, (D^*/v^*) , declines as the corporate tax rate increases, when bankruptcy costs are low. Optimal debt, D^* , increases with τ , but (unlike the unprotected debt case) increases less rapidly than v^* .

VI. Discussion and Variations: Debt Value and Capital Structure

Our analysis has determined optimal leverage ratios and associated yield spreads in a variety of environments, for both long-term unprotected debt

Table III
Comparative Statics of Financial Variables at the Optimal Leverage Ratio: Protected Debt

This table describes the behavior of the coupon, C^* , that maximizes firm value, and the debt, D^* , leverage, L^* , interest rate, R^* , yield spread, $R^* - r$, total firm value, v^* , equity value, E^* , and bankruptcy value, V_B^* , at the optimal coupon level, for debt protected by a positive net-worth covenant. V is the firm's asset value, σ^2 is the variance of asset returns, r is the risk-free interest rate, α is the fraction of asset value lost if bankruptcy occurs, and τ is the corporate tax rate.

Variable	Shape	Sign of Change in Variable for an Increase in:			
		σ^2	r	α	τ
C^*	Linear in V	$< 0^b$	> 0	< 0	$> 0^a$
D^*	Linear in V	< 0	> 0	< 0	$> 0^a$
L^*	Invariant to V	< 0	> 0	< 0	$< 0, \alpha$ small ^b ; $> 0, \alpha$ large
R^*	Invariant to V	$> 0^a$	> 0	$> 0^b$	$> 0^a$
$R^* - r$	Invariant to V	$> 0^a$	$> 0^b$	$> 0^b$	$> 0^a$
v^*	Linear in V	< 0	> 0	< 0	> 0
E^*	Linear in V	> 0	< 0	> 0	$> 0, \alpha$ small ^b ; $< 0, \alpha$ large
V_B^*	Linear in V	< 0	> 0	< 0	$> 0^a$

^aNo effect if $\alpha = 0$.

^bRepresents different behavior from unprotected debt.

and protected (or continuously rolled-over) debt. It is of interest to compare these results with typical leverage ratios and yield spreads in the United States. Leverage in companies with highly rated debt is generally less than 40 percent. Yields of investment-grade corporate bonds have exceeded Treasury bond yields by a minimum of 15 basis points (bps), and a maximum of 215 bps from 1926 to 1986. The average yield spread over this period was 77 bps.³² These spreads reflect finite-maturity debt and also reflect the fact that corporate debt typically is callable. Call provisions may add about 25 bps to the annual cost of corporate debt.³³ Subtracting 25 bps from the average yield spread of 77 bps to eliminate the impact of call provisions gives an adjusted historical yield spread of about 52 bps.

We examine a base case where the volatility of the firm's assets is 20 percent, the corporate tax rate is 35 percent, the risk-free rate is 6 percent, and bankruptcy costs are 50 percent. In this case, optimal leverage with unprotected debt is 75 percent and the yield spread is 75 bps. The optimally

³²As reported by Kim, Ramaswamy, and Sundaresan (1993); see also Sarig and Warga (1989).

³³Kim, Ramaswamy, and Sundaresan (1993) estimate a call premium of 22 bps using a numerical example.

levered firm's equity is volatile, with a 57 percent annual standard deviation. Reducing the effective tax rate would reduce optimal leverage and the yield spread. For example, with an effective tax rate of 15 percent, optimal leverage is 59 percent, and the yield spread is 35 bps. Equity volatility is lower, but still substantial.

It is clear that, based on our assumptions thus far, the analysis of unprotected debt suggests optimal leverage considerably in excess of current practice. This could be construed as a criticism of current management rather than the model. Managers may be loath to pay out "free cash flow" (see Jensen (1986)); the wave of leveraged buyouts in the late 1980s suggests that firm value may be raised by using greater leverage (see Kaplan (1989) and Leland (1989)). However, the model's predicted yield spreads seem low given the suggested high leverage.

Optimal leverage ratios and yield spreads for protected debt are more consistent with historical ratios. In the base case, optimal leverage is 45 percent and the yield spread is 45 bps. Equity has a 34 percent annual standard deviation, which is a bit higher than the historical average equity risk of a single firm of about 30 percent.

We now consider how variations in the assumptions may affect the nature of optimal leverage and yield spreads, in both the unprotected and protected cases.

A. No Tax Shelter for Interest Payments When Value Falls

We have assumed that the deductibility of interest payments generates tax savings at all values above the bankruptcy level. But as firm asset value drops, it is quite possible that profits will be less than the coupon payout and tax savings will not be fully realized (or will be substantially postponed). If lesser tax benefits are available, the optimal leverage ratio declines.

In Appendix A, we extend the analysis to allow for no tax benefits whenever $V < V_T$, where V_T is an exogenously specified level of firm asset value.³⁴ In the base case considered above, optimal leverage falls from 75 to 70 percent, and the yield spread at the optimal leverage rises from 75 to 87 bps, when $V_T = 90$, i.e., 90 percent of the current asset value.

A possible criticism of the above approach is that V_T does not depend upon the amount of debt the firm has issued. Consider an alternative scenario in which higher profit is needed if higher coupon payments are to be fully deductible. For example, assume that the rate of EBIT is related to asset value as follows:

$$\text{EBIT} = (V - 60)/6. \quad (30)$$

Thus gross profit before interest drops to zero when V falls to 60 and represents one-sixth of asset value in excess of 60. Further, assume that

³⁴ We do not allow tax loss carryforwards in this analysis, since they would introduce a form of time (and path) dependence. Thus, we may overstate the loss of tax shields: the "truth" perhaps lies somewhere between the previous results and the results of this analysis.

coupon payments, C , can be deducted from profit (for tax purposes) only if $EBIT - C \geq 0$. (We ignore partial deductibility.) It then follows that

$$V_T = 60 + 6C. \tag{31}$$

In contrast with the previous scenario, greater debt now has a greater likelihood of losing its tax benefits.³⁵ Optimal leverage falls to 65 percent. The yield spread falls to 61 bps, reflecting the lesser leverage. Equity volatility remains high at 51 percent. In the case of protected debt, the loss of tax deductibility has a much smaller effect on optimal leverage and yield spread. As expected, the loss of tax deductibility reduces the maximum value of the firm in all cases.

B. Net Cash Payouts by the Firm

Following Brennan and Schwartz (1978) and others, we have focused on the case where the firm has no net cash outflows resulting from payments to bondholders or stockholders. We now change this assumption.³⁶ Net cash outflows may occur because dividends are paid to shareholders, and/or because after-tax coupon expenses are being paid, without fully offsetting equity financing. In this latter case, assets are being liquidated and the scale of the firm’s activities is clearly affected by the extent of debt financing.

To keep matters analytically tractable, we consider only cash outflows that are proportional to firm asset value, where the proportion, d , may depend on the coupon paid on debt. Equation (3) is replaced by

$$(1/2)\sigma^2 V^2 F_{VV}(V) + (r - d)VF_V(V) - rF(V) + C = 0, \tag{32}$$

with general solution

$$F(V, t) = A_0 + A_1 V^{-Y} + A_2 V^{-X}, \tag{33}$$

where

$$X = \left\{ (r - d - 0.5\sigma^2) + \left[(r - d - 0.5\sigma^2)^2 + 2\sigma^2 r \right]^{1/2} \right\} / \sigma^2 \tag{34}$$

$$Y = \left\{ (r - d - 0.5\sigma^2) - \left[(r - d - 0.5\sigma^2)^2 + 2\sigma^2 r \right]^{1/2} \right\} / \sigma^2. \tag{35}$$

Boundary conditions remain unchanged, implying $A_1 = 0$ as before since $Y \leq -1$. Therefore, solutions for all security values will have exactly the same functional form as before, but with the exponent, X , given by equation (34) rather than equation (5).

When $d = 0.01$, a one-percent payout on asset value (equivalent to approximately a 3 percent dividend on equity value, given the leverage of the base

³⁵In the base case the optimal coupon falls to \$5.08, implying V_T is about 90, as above.

³⁶The reader may wonder how equity value could be positive if the firm never pays dividends. But our earlier assumption is *not* that firms never pay dividends—rather, there is no net cash outflow: any cash dividends must be financed by issuing new equity. Like Black and Scholes (1973), our model is a partial equilibrium one, and simply assumes the process for V .

case), the optimal leverage falls from 75 to 74 percent, and the yield spread rises from 75 to 86 bps. But what if payouts also depend upon the coupon being paid to debtholders? Consider the case where the proportional payout is sufficient to cover the after-tax cost of debt when it is initially offered.³⁷ Normalizing the initial value of V to 100 implies a payout $d = (1 - \tau)C/100$, or $0.0065C$ in the above example. Any dividend payout would be in addition to this amount. For the base case above, we search over coupon levels, C , that maximize v , subject to the constraint that $d = 0.0065C + 0.01$. This reduces optimal leverage from 75 to 64 percent and increases the yield spread at the optimum from 75 to 124 bps. The volatility of equity falls from 57 to 42 percent. In the case of protected debt, optimal leverage falls from 45 to 36 percent, the yield spread increases from 45 to 49 bps, and the volatility of equity falls from 34 to 29 percent.

The maximum firm value drops from \$128.4 to \$122.0 with unprotected debt, and from \$113.3 to \$110.0 with protected debt. This decrease in maximal value reflects the fact that bankruptcy is more likely with cash payouts, with a resulting loss of tax benefits. Therefore, *ex ante*, *shareholders (as well as bondholders) benefit from a covenant that prevents the firm from selling assets to meet coupon payments*. It is not surprising that many debt instruments have such a preventive covenant. But if such a covenant cannot be written (or cannot be enforced), shareholders will benefit (at bondholders' expense) from the firm selling assets to pay coupons *after* the debt has been issued. Recognizing this incentive, debtholders will pay less for debt and the optimal leverage will fall as indicated above.

C. Absolute Priority Not Respected

We have assumed that debtholders receive all assets (after costs) if bankruptcy occurs, and stockholders none: the "absolute priority" rule. Now consider a simple alternative, where debtholders receive some fraction $(1 - b)$ of remaining assets, $(1 - \alpha)V_B$, while equity holders receive $b(1 - \alpha)V_B$.³⁸ This will affect debt value in two ways: debtholders will receive less value if bankruptcy occurs, and bankruptcy will occur at a different level V_B .

It can readily be shown that equation (7) will be replaced by

$$D(V) = C/r + [(1 - b)(1 - \alpha)V_B - C/r][V/V_B]^{-X}, \quad (36)$$

³⁷Note that as value falls, the proportional payout will no longer completely cover the after-tax coupon—some equity financing becomes necessary. This may not be unreasonable, since bondholders will become increasingly sensitive to liquidation of assets as firm value approaches the bankruptcy level.

³⁸Franks and Torous (1989) estimate that deviations in favor of equity holders in Chapter 11 reorganizations are only 2.3 percent of the value of the reorganized firm. Eberhart, Moore, and Roenfeldt (1990) estimate average equity deviations of 7.8 percent for their sample of Chapter 11 firms. We choose a 10 percent deviation as an upper bound for this effect.

and equation (14) will be replaced by

$$V_B = (1 - \tau)C/[r(1 - b + \alpha b)][X/(1 + X)]. \quad (37)$$

For the base case with unprotected debt, deviations from absolute priority of 10 percent ($b = 0.1$) cause the optimal leverage ratio to fall from 75 to 72 percent. The yield spread remains at 75 bps. The effect of deviations from absolute priority are also minor when debt is protected: leverage remains unchanged at 45 percent, while the yield spread rises from 45 to 51 bps.

D. All of the Above

As a final exercise, consider the base case where, in addition, (i) dividends equal 3 percent of equity value; (ii) after-tax coupon payments are *not* initially financed with additional equity; (iii) coupon payments are not tax deductible when $V < V_T = 60 + 6C$; and (iv) there is a 10 percent deviation from absolute priority ($b = 0.1$). When these conditions hold simultaneously, the optimal leverage with unprotected debt falls to 47 percent and the yield spread is 69 bps. The annual standard deviation of equity is 36 percent. For protected debt, the optimal leverage falls to 32 percent, the yield spread is 52 bps and the standard deviation of equity is 29 percent. These last numbers seem quite in line with historical yield spreads, leverage ratios, and equity risks.

VII. Protected versus Unprotected Debt: Potential Agency Problems

Our results suggest that optimal leverage ratios are lower when debt is protected, and that the maximal gains to leverage are less. This raises a key question: why should firms issue protected debt? The answer may lie with agency problems created by debt, and asset substitution in particular. Jensen and Meckling (1976) argue that equity holders would prefer to make the firm's activities riskier, *ceteris paribus*, so as to increase equity value at the expense of debt value. The expected cost to debtholders will be passed back to equity holders in a rational expectations equilibrium, through lower prices on newly issued debt.

Higher firm asset risk tends to benefit equity holders when equity is a strictly convex function of firm asset value, V . And equity is strictly convex in V when debt is unprotected. In Section V, however, it was shown that equity may be a strictly *concave* function of V when debt has a positive net-worth covenant. With protected debt, stockholders may *not* have an incentive to increase firm risk at debtholders' expense.

To illustrate our point, consider the base case above with different levels of asset volatility. If debt is unprotected, the optimal coupon is \$6.50, firm value is \$128.4, and V_B is \$52.8. If debt is protected, the optimal coupon is \$3.26, firm value is \$113.3, and V_B is \$50.6. Assume that, *ex post*, managers can raise the risk of the firm's assets from the current annual standard deviation

Table IV
Values of Protected and Unprotected Debt and Equity for
Different Levels of Risk

This table gives the values of debt and equity for both unprotected and protected debt, when the coupon (in each case) is chosen to maximize total firm value given a 20 percent asset volatility, but asset volatility may be increased by management to higher levels.

Asset Volatility (%)	Unprotected Debt		Protected Debt	
	Debt Value (\$)	Equity Value (\$)	Debt Value (\$)	Equity Value (\$)
20	96.3	32.1	50.6	62.7
40	70.4	45.9	36.9	55.5
60	52.6	59.1	31.2	52.5

of 20 percent—with no change in current asset value V . Will they be motivated to engage in such “asset substitution”? Using equations (7) and (13), and recalling from equation (14) that V_B will change when debt is unprotected but not when debt is protected, gives the results reported in Table IV.

Debtholders are hurt by higher risk. In the case of unprotected debt, equity value is enhanced by greater risk. Without covenants to prevent such a change, it will always be in the interest of equityholders to increase risk. *But the opposite is true when debt is protected by a positive net worth covenant:* in this case, increasing risk lowers equity value as well as debt value.³⁹

In the absence of protective covenants, investors recognize that shareholders will wish to raise asset volatility to the maximum (60 percent). They will pay only \$52.6 for the debt, and total firm value will be \$111.7. If the firm offers protected debt, investors recognize that shareholders will have no interest in increasing firm risk, and total firm value will be \$113.30. The firm maximizes value by issuing protected rather than unprotected debt. (Even if the firm initially chose to issue the amount of unprotected debt optimal for a 60 percent volatility, the total firm value would be \$112.1—less than the maximal value with protected debt.)

A reevaluation of the belief that asset substitution is always advantageous for equity holders seems warranted. It is true for unprotected debt, but it is false in the case examined here, when debt is protected by a positive net-worth covenant. Both debt and equity are concave functions of asset value

³⁹The difference in behavior as σ changes reflects the convexity (concavity) of equity in V when debt is unprotected (protected). In addition, V_B changes in the unprotected case as σ changes. This latter effect explains the curious result in Section II.A, that (for V close to V_B) an increase in firm risk can raise unprotected debt value. Thus, there is yet a further anomaly with unprotected debt: at the brink of bankruptcy (and only there), both debtholders and stockholders wish to increase firm risk!

in this case.⁴⁰ The greater incentive compatibility of protected debt may well explain its prevalence (or the prevalence of short-term financing), despite the fact that, *ceteris paribus*, it exploits the tax advantage of debt less effectively.

VIII. Restructuring via Debt Repurchase or Debt Renegotiation: Some Preliminary Thoughts

The preceding analysis has assumed that the coupon, C , of the debt issue is fixed through time. In the absence of transactions costs, restructuring by continuous readjustments of C would seem to be desirable to maximize total firm value as V fluctuates. However, we shall see that continuous readjustments of C by debt repurchase (issuance) may be blocked by stockholders (debtholders).⁴¹ *Debt renegotiation* may be required to maximize total firm value in these cases.

To prove this contention, first consider the firm selling a small amount of additional debt, thereby increasing the current debt service by dC . This will change the total value of debt by

$$dD = (\partial D / \partial C) dC. \tag{38}$$

But this total value change will be shared by current and new debtholders. New debtholders will hold a fraction dC/C of the total debt value, leaving current debtholders with value

$$(D + dD)(1 - dC/C) = D + dD - (D/C)dC, \tag{39}$$

(ignoring terms of $O(dC^2)$). The change from D , the current debtholders' value before the debt issuance, is

$$[(\partial D / \partial C) - (D/C)]dC < 0 \quad \text{for } dC > 0, \tag{40}$$

with the inequality resulting from the concavity of D in C and the fact that $D = 0$ when $C = 0$. This "dilution" result holds for arbitrary initial V and C , implying current *debtholders will always resist increasing the total coupon payments through additional debt issuance*, even though such sales may increase the value of equity and the firm. This resistance is frequently codified in debt covenants that restrict additional debt issuance at greater or equal seniority.⁴²

⁴⁰This, of course, is consistent with the earlier result, equation (12), where total firm value, v , is concave in V . Concavity of v follows from the concavity of tax benefits and convexity of bankruptcy costs (which are subtracted).

⁴¹We consider debt issuance/repurchase for capital restructuring only. Any funds raised by debt issuance will be used to retire equity, and vice-versa. Debt raised for new investment, or retired by asset sales, are asset-changing decisions that are not considered here.

⁴²Our analysis assumes a single class of debt, implying that newly issued debt has the same seniority in the event of bankruptcy. Even if the newly issued debt is junior to the current debt, it will reduce the value of the current debt by raising V_B . A full analysis of multiple classes of debt securities is beyond the scope of the present article.

A related result on debt repurchase is perhaps more surprising: current shareholders will always resist *decreasing* the coupon, C , by repurchasing current debt (in small amounts) on the open market. To prove this, consider a small decrease, $dC < 0$, and its effect on current shareholders. The total value of equity will change by

$$dE = (\partial E / \partial C) dC \quad (41)$$

The cost of retiring debt will equal the value of the fraction of debt retired, or $-D(dC/C)$. This cost must be financed with newly issued equity, whose value is included in the change in total equity value above. Current shareholders will therefore have equity value

$$E + dE + D(dC/C), \quad (42)$$

implying a change in value to current shareholders of

$$[(\partial E / \partial C) + (D/C)] dC, \quad (43)$$

using equation (41). For unprotected debt, it follows from equations (15) and (17) that $[(\partial E / \partial C) + (D/C)] > 0$, implying that the change in equity value to the original shareholders is negative when $dC < 0$. This result holds for arbitrary initial V and C . Therefore, it will never be optimal for the firm's shareholders to restructure by retiring unprotected debt via small open market repurchases financed by new equity.⁴³ In Appendix B, we show that the result also holds for small repurchases of protected debt, when the coupon is near its optimal level.

To illustrate the arguments above, consider the base case with unprotected debt. With $V = \$100$, the optimal coupon is $\$6.50$ and $V_B = \$52.80$. Assume this coupon level has been chosen by the firm. Now let V drop from $\$100$ to $\$90$. Using equations (7) and (13) to compute the current values of debt and equity gives: $D = \$91.79$, $E = \$23.14$, and $v = \$114.93$. The firm's total value can now be increased by reducing debt. The firm should cut its coupon by 10 percent to $\$5.85$, since C^* is proportional to V , which has fallen from $\$100$ to $\$90$. This would increase the total firm value from $\$114.93$ to $\$115.60$.

But consider the firm repurchasing 10 percent of its debt to achieve the new optimal leverage. The coupon is reduced from $\$6.50$ to $\$5.85$, and V_B falls by 10 percent to $\$47.52$. The firm must pay (at least) $\$9.18$ to retire 10 percent of the bonds whose value is $\$91.79$ prior to repurchase.⁴⁴ It will raise this amount by issuing additional stock worth $\$9.18$. Again using equations (7) and (13) to compute debt and equity values with the lower coupon gives: $D = \$86.65$, and $E = \$28.95$.

⁴³This debt repurchase result holds even if there are multiple classes of debt. Stockholders might benefit from retiring debt via asset sales, but this would violate the assumption that the asset value, V , is independent of the firm's capital structure.

⁴⁴Note that debt becomes more valuable per unit, as the coupon is reduced. We are assuming here that the entire amount of repurchase can be effected at the lowest (i.e., current) price. Any higher price would magnify the losses to equity holders.

Debt holders are clearly better off, having received payments of \$9.18 to retire 10 percent of their holdings, plus retaining holdings worth \$86.65. The original equity holders have had their stock diluted: \$9.18 of stock—the amount raised to pay the debtholders—now belongs to new shareholders, leaving the original shareholders with stock worth $\$28.95 - \$9.18 = \$19.77$. This is less than the \$23.14 value of their shares prior to repurchase. Although the total value of the firm would be increased by the restructuring, *equity holders cannot benefit from the repurchase, and will want to block such refinancing*. This problem results from an externality: when debt is reduced, its “quality” is improved. Investors who continue to hold the firm’s debt receive a windfall gain from the debt repurchase.

The example shows that restructuring through debt repurchases or sales may not be possible, although such changes could increase total firm value. To capture such potential increases, changes in the terms of the debtholders’ securities (or “side payments”) will be required. These types of restructurings will be labeled *debt renegotiation*. In our example, replacing current debt with convertible debt may be used to achieve the optimal coupon level. By agreeing to exchange the current debt for debt with coupon \$5.85 (worth \$86.65), plus a convertibility privilege into stock worth (say) \$5.50, debtholders receive a security worth \$92.15. This exceeds the \$91.79 value of the current debt paying a \$6.50 coupon, so bondholders will benefit. Stockholders will also benefit by the rise in the equity value of \$5.81 ($\$28.95 - \23.14) less the \$5.50 value of the convertibility option given bondholders.

Renegotiation of unprotected debt is particularly simple when bankruptcy is imminent (V is close to V_B), and $C > C_{max}(V)$. In this case, a small reduction in the coupon will increase the value of both debt and equity—with no further compensation to bondholders (such as the convertibility privilege) being required. The firm may be able to reduce its coupon payment all the way to $C^*(V)$ with no additional payments to bondholders if the value of debt $D^*(V)$ at the optimal coupon is greater than the value of debt $D(V)$ when the renegotiation begins. This assumes stockholders can credibly make a “take-it-or-leave-it” offer to bondholders. Note that the firm may wait until the brink of bankruptcy before renegotiating, since this will minimize $D(V)$.

IX. Conclusion

By assuming a debt structure with time-independent payouts, we have been able to develop closed-form solutions for the value of debt and for optimal capital structure. This permits a detailed analysis of the behavior of bond prices and optimal debt-equity ratios as firm asset value, risk, taxes, interest rates, bond covenants, payout rates, and bankruptcy costs change.

The analysis examines two types of bonds: those that are protected by a positive net-worth covenant, and those that are not. The distinction is critical in determining when bankruptcy is triggered, which in turn affects bond

values and optimal leverage. To be rolled over, short-term financing typically requires that the firm maintain positive net worth. Therefore short-term financing seems to correspond to our model of protected debt. Long-term debt, in contrast, rarely has positive net-worth covenants; it seems closer to our model of unprotected debt.

Our results indicate that protected debt values and unprotected "investment grade" debt values behave very much as expected. Unprotected "junk" bonds exhibit quite different behavior. For example, an increase in firm risk will increase debt value, as will a decrease in the coupon. Such behavior is not exhibited by protected "junk" bonds.

Two curious aspects of optimal leverage are observed. First, a rise in the risk-free interest rate (increasing the cost of debt financing) leads to a *greater* optimal debt level. Higher interest rates generate greater tax benefits, which in turn dictate more debt despite its higher cost. Second, the optimal debt for firms with higher bankruptcy costs may carry a *lower* interest rate than for firms with lower bankruptcy costs. This is because firms will choose significantly lower optimal leverage when bankruptcy costs are substantial, making debt less risky. This result does not hold for protected debt: higher bankruptcy costs imply higher interest rates at the optimal leverage.

Optimal leverage, yield ratios, and equity risk are well within historical norms for protected debt. But optimal leverage seems high (and/or yield spreads seem low) for unprotected debt. Variants of the basic assumptions, discussed in Section VI, are needed for unprotected debt to fall within historical norms. The most important modification is dropping the requirement that payouts to bondholders be externally financed.

Issuing debt without protective net-worth covenants yields greater tax benefits and would seem to dominate issuing protected debt. However, this conclusion may be reversed if firms have the ability to increase the riskiness of their activities through "asset substitution." Increasing risk will transfer value from bondholders to stockholders when debt is unprotected, leading cautious bondholders to demand higher interest rates even when the firm currently has low risk. But such costs typically are not incurred when firms issue protected debt: stockholders will *not* gain by increasing firm risk when debt is protected by a positive net-worth covenant, and bondholders will not need to demand higher interest rates in anticipation of riskier firm activities. Protected debt may be the preferred form of financing in these situations, despite having lower potential tax benefits.

Our results offer some preliminary insights on debt repurchases and on debt renegotiations. The former cannot be used to adjust leverage continuously to its optimal level: bondholders will block further debt issuance, and shareholders will block (marginal) debt reductions. Debt renegotiation can achieve simultaneous increases in debt and equity value. But the costly nature of renegotiation suggests it would be suboptimal to do so continuously (see Fischer, Heinkel, and Zechner (1989)). Our analysis shows that it may be desirable for shareholders to wait until the brink of bankruptcy before renegotiating. When bankruptcy is neared, a reduction in coupon payments to

the optimal level may benefit both stockholders and bondholders, without additional side payments.

Although we have not emphasized equity values, our analysis also provides some interesting insights. Equity return volatility will be stochastic, changing with the level of firm asset value, V . This (and the possibility of bankruptcy) has important ramifications for option pricing.⁴⁵

The model can be extended in several further dimensions. Multiple classes of long-term debt can be analyzed, recognizing that payments to the various classes of debtholders when bankruptcy occurs are determined by seniority. More difficult extensions will include finite-lived debt, dynamic restructuring, and a stochastic term structure of risk-free interest rates.

Appendix A

We assume in this case that instantaneous tax benefits = 0 whenever $V \leq V_T$. There are no carryforwards. Differential equation (3) with $C = 0$ has solution:

$$TB(V) = A_1V + A_2V^{-X}, \quad V_B < V \leq V_T. \quad (44)$$

Differential equation (3) with instantaneous tax benefit τC realized has solution:

$$TB(V) = (\tau C/r) + B_2V^{-X}, \quad V \geq V_T. \quad (45)$$

$TB(V)$ must satisfy:

$$TB[V_B] = A_1V_B + A_2V_B^{-X} = 0 \quad (46)$$

$$TB(V_T) = A_1V_T + A_2V_T^{-X} = (\tau C/r) + B_2V_T^{-X} \quad (47)$$

$$TB'(V_T) = A_1 - XA_2V_T^{-X-1} = -XB_2V_T^{-X-1} \quad (48)$$

Solutions:

$$A_1 = (\tau C/r)(X/(X + 1))(1/V_T) \quad (49)$$

$$A_2 = -(\tau C/r)(X/(X + 1))(V_B^{X+1}/V_T) \quad (50)$$

$$B_2 = -(\tau C/r)(X/(X + 1))(1/V_T)(V_B^{X+1} + (1/X)V_T^{X+1}) \quad (51)$$

Substituting for tax benefits from equation (44) into equation (13) for equity gives, for $V \leq V_T$,

$$E = v - D = V + A_1V + A_2V^{-X} - C/r - [V_B - (C/r)](V/V_B)^{-X}. \quad (52)$$

To find V_B , we again set $dE/dV|_{V=V_B} = 0$:

dE/dV

$$= 1 + A_1 - XA_2V^{-X-1} + X[V_B - (C/r)](V/V_B)^{-X-1}(1/V_B) = 0. \quad (53)$$

⁴⁵Preliminary work on this question has been done by Klaus Toft (1993).

Evaluating equation (53) at $V = V_B$:

$$1 + A_1 - XA_2V_B^{-X-1} + X - (C/r)(X/V_B) = 0. \quad (54)$$

Substituting for A_1 and A_2 gives

$$V_B = CV_T X / [rV_T(1 + X) + \tau CX]. \quad (55)$$

D can be computed from equation (7); and

$$v = V + (\tau C/r) + B_2V^{-X} - \alpha V_B(V/V_B)^{-X}, \quad V > V_T. \quad (56)$$

Note we can rewrite the expression for V_B as

$$V_B = \underline{V}_B V_T / [(1 - \tau)V_T + \tau \underline{V}_B] > \underline{V}_B \quad (57)$$

with the last inequality holding since $V_T > \underline{V}_B$, where \underline{V}_B satisfies equation (14).

Appendix B

Parallel to the discussion following equation (43), we know that shareholders will reject a buyback of debt (i.e., $dC < 0$) if $[(\partial E/\partial C) + (D/C)] > 0$. Since $E = v - D$,

$$[(\partial E/\partial C) + (D/C)] = \{\partial v/\partial C - [(\partial D/\partial C) - (D/C)]\}. \quad (58)$$

Define V^* as the firm asset value at which the current coupon would be optimal for protected debt, i.e., for which $\partial v(V^*)/\partial C = 0$. From equation (40), it follows that $[(\partial D/\partial C) - (D/C)]$ will be strictly negative, and therefore equation (58) will be strictly positive when $\partial v/\partial C = 0$. Continuity implies that there exists a neighborhood of values, V , around the value V^* , for which equation (58) is strictly positive. For all $V < V^*$ in this neighborhood, firm value, v , would be increased by lowering the coupon, since the optimal coupon is decreasing in V . But because equation (58) is positive in this neighborhood, current stockholders' equity value will fall when $dC < 0$ and shareholders will resist reducing the coupon to its optimal level.

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